Rhythm, Resonance and Harmony

The mathematics of music

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Preface

Music is the pleasure that the human mind experiences from counting without being aware that it is counting.
Gottfried Wilhelm von Leibniz

The latest music being composed is nothing if not diverse, making use of computers to make melodies and beats out of almost anything. Mathematics, electronics, bits and bytes seem to be made for music, accompanying it to new frontiers. Was music more bland at the start of the 20th century? In the 5th century? And 1,000 years before Christ? Were mathematics used to study sounds in those ancient days? When did technology start to influence music?

Music is one of humanity's chief means of expression. It is made everywhere and is a constant feature of cultural histories. It is always there, ready to move people, making them happy one minute and sad the next. Mathematics analyses this phenomenon and the results illuminate an endless number of aspects of music: the ratios between the sounds of a chord, the phenomenon of resonance, the secret keys to the score, musical games, the geometric structures of melodies. Those who know how to enjoy mathematics can add to the pleasure they experience as listeners with the surprises revealed by the mathematical ingredients.

We shall see how Mozart formulated a method for composing music using dice. Some works are designed to only make sense when a mathematical riddle is solved. Chance, fractals and the golden ratio also have their place on the stave. Why are some sounds dissonant and others harmonic? What makes it possible to distinguish between the sound of a violin or a trumpet? Can a singer shatter a glass with their voice alone? What contributions has technology made to music? How did modern musical notation evolve and what rules does it follow?

If music is infused with the most diverse forms of mathematics, it is important to point out that science does not fully explain the phenomenon of music. It does, however, provide a wide range of tools for creating it, and we also take a closer look at that in this book. At any rate, with or without mathematical tools, the key to any work of music continues to be in the inspiration and art of the composer. And therein, ultimately, lies the value that mathematics offers music – frameworks for contemplating and admiring an artistic form. These are new perspectives from which we can rediscover what we thought we already knew.

Chapter 1

Tuning Strings

After silence, that which comes nearest to expressing the inexpressible is music.
Aldous Huxley

A musical performance is ephemeral – once a tone or chord is played it only really exists in the memories of the listeners. This property gives music a magical aura, and humankind has used it in its rituals from the beginning of time, using it as a means to worship and commune with the gods. Archaelogical finds reveal that musical instruments were already being used throughout the length and breadth of the world in prehistoric times. They were generally rattles and other percussion instruments, although primitive flutes and pipes have also been found, testifying to the presence of melody in the music of the oldest cultures.

The Greek world

The word 'music' has its roots in the Greek word *musiké,* literally, 'the art of the muses'. In Greek mythology, muses were gods of inspiration for music, dance, astrology and poetry.

The Pythagorean school, active from the 6th century BC, tried to explain the harmony of the universe with numbers – and music was very much part of this field of study. The Pythagoreans devised astronomical, acoustic and musical models that allowed music and arithmetic to be studied together as two sides of the same coin. They believed that the movements of the planets in space created harmonic vibrations that were imperceptible to the human ear but produced "the music of the spheres" nevertheless.

Generally speaking, Graeco-Roman civilisations developed theoretical knowledge separately from manual activities, which were known as 'lesser arts'. The higher disciplines were brought together in two large groups: the first, named *trivium* (from *tri,* 'three', and *vium,* 'way' or 'path'), was made up of grammar, dialectics and rhetoric; the second, named *quadrivium* (from *quadri,* 'four'), was composed of

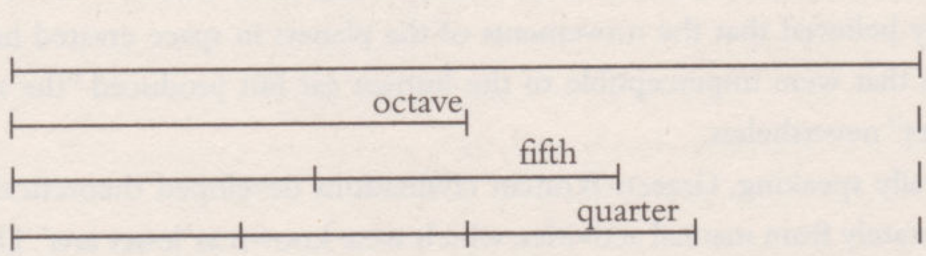
arithmetic, geometry, astronomy and music. These were the seven paths, or 'liberal arts' that ensured a man remained in equilibrium with the harmonious universe.

The Pythagorean musical system

The Pythagorean study of music was based on the sounds produced when strumming a single stringed instrument. The length of the string was changed in a similar way to a modern guitar, which uses a fretboard. Varying the length of the string created different musical notes or tones. The shorter the string, the higher the pitch of the note. The Pythagoreans made a methodical comparison of the pairs of sounds produced by the various string lengths. Their experiments involved ratios of the lengths expressed using small numbers – dividing the string in half, one third and two thirds of the original length, etc.

The results were surprising: the sounds created by the strings with lengths that were related by small numbers resulted in more pleasant sounds, or rather these were the most harmonious to the ear. Thanks to these observations, it was possible for the Pythagoreans to mathematically model a physical phenomenon in relation to its aesthetics in a similar way to the role played by the golden ratio in the optical investigation of beauty during the Renaissance.

The simplest ratio is when the string is held down at half its length. This ratio is numerically expressed as 2:1, and musically corresponds to an octave interval (e.g. the distance between one *C* and the following *C*). The next simplest ratio is when the string is held down at a point located at one third of its total length; expressed numerically, the ratio is 3:2, which corresponds to the interval of a fifth (the distance *C* to *G*). Next we have the ratio where the string is held down at a quarter of the total length, numerically expressed by the ratio 4:3, and in music terms, this corresponds to the interval of a fourth (the distance *C* to *F*).



PLANETARY SOUNDS

The idea of the cosmos in a state of equilibrium is a notion that Renaissance humanism sought to recover from classical culture. One expression of this equilibrium, as defined by the Pythagoreans and similarly expressed by Plato and Aristotle, was the "harmony of the spheres". This was based on the idea that the planets generated sounds inaudible to the human ear in proportion to their position and movement, and furthermore, that these sounds were consonant, or harmonic. The German Johannes Kepler (1571–1630) had studied religion, ethics, dialectics and rhetoric, as well as physics and astronomy. He also became interested in the heliocentric theory of the Universe and the legacies of Pythagoras and Plato. At the start of the 17th century, planetary motion was still a mystery that could only be explained by the will of God. Kepler shed light on the mystery, and his laws of planetary motion rank among some of the great scientific discoveries of all time. However his theories went further, seeking to recover a vision of the harmony of the spheres from the classical world. Thus in *Harmonices Mundi* (*The Harmony of the Worlds*), written in 1619, Kepler explained, alongside his astronomical studies, the theory that each planet emitted a sound that was determined by its angular velocity. The limits of this angular velocity were at the perihelion (the closest point to the Sun) and the aphelion (the furthest point from the Sun) of its elliptic path. Kepler compared the sounds at these limit points for a given planet and also between neighbouring planets. This led him to establish scales and chords associated with each of them. According to his calculations, the melodies of Venus and the Earth varied by an interval of one semitone or less. At the other extreme, Mercury's maximum interval was greater than an octave. Kepler's religious views led him to speculate that the planets had been in harmony on very few occasions; perhaps, he thought, only at the moment of Creation.

An image from Kepler's Harmonices Mundi *illustrating the sounds that were supposedly produced by the planets.*

PYTHAGORAS OF SAMOS (582 BC–496 BC)

Pythagoras was born on the Greek island of Samos. Inspired by the philosopher and mathematician Thales of Miletus, he began a extensive journey of apprenticeship throughout Egypt and Mesopotamia. His experiences aroused ideas that led him to found a school of thought in which various scientific, aesthetic and philosophical disciplines coexisted. Pythagoras and his followers undertook important studies in areas as diverse as acoustics and music, mathematics, geometry and astronomy. The prestige of Pythagoras and his school was such that one of the fundamental theorems in geometry is often attributed to it, namely Pythagoras' theorem, even if it was previously known by the many advanced civilisations centuries before. Pythagoras' theorem can be summarised by the following formula

$$a^2 + b^2 = c^2$$

which gives us an equation with infinite whole number solutions. Each set of solutions is referred to as a 'Pythagorean triplet'. When drawn, each set of three elements that determines a Pythagorean triplet gives a set square, an instrument used by farmers and craftspeople to draw right angles.

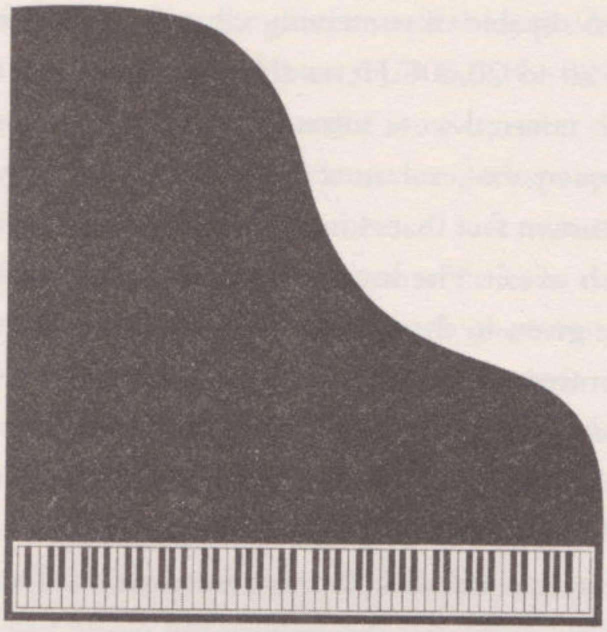
Thus we can see the emergence of a pattern in which the intervals of sounds with ratios of the type

$$\frac{n+1}{n}$$

are harmonic and melodious. The Pythagoreans interpreted this as confirmation of the direct connection between the number, harmony and beauty.

Absolute tuning

In order to better appreciate the power of the Pythagorean discoveries, it is helpful to distinguish between two key concepts: 'absolute' tuning and 'relative' tuning. Each musical note has a pitch, which identifies it as lower or higher than another. The pitch of the note is determined by the frequency of the oscillation of its sound wave (we shall return to this point later on). Higher frequencies result in higher pitched sounds. (Appendix I provides a detailed explanation of these and other musical concepts.)



On a piano, the keys corresponding to the lowest notes are on the left, while the highest notes are on the right.

SHATTERING GLASSES AND COLLAPSING BRIDGES

In a number of films and cartoons, we often see an opera singer shattering a wine glass by emitting a high-pitched sound. This is not fictitious physics but is absolutely real. Rigid bodies can vibrate at a given 'natural' frequency that is determined by the material of which they are composed, their shape and other properties. In addition to this, there is a sound source that produces a noise that reaches the object in the form of fluctuating waves of air pressure, causing it to vibrate. However. when the frequency of the sound which is emitted is the same as the natural frequency of the object, the solid begins to vibrate with greater intensity in a phenomenon called 'resonance'. If the phenomenon of resonance is added to an increase in the energy (volume) of the sound source, the amplitude of the vibrations of the body increases further. In the case of a rope, the material's own flexibility allows it to withstand these variations. If, on the other hand, we are dealing with a highly rigid body, it will be unable to absorb the vibrations and will end up shattering. This is what happens to the wine glass. There is a famous example, the Tacoma Narrows suspension bridge that collapsed on 7 November 1940, a few months after it was opened. The bridge collapsed above Puget Sound in the US state of Washington due to oscillations caused by the wind, a phenomenon known as 'flutter'.

The human ear is capable of perceiving vibrations with frequencies in an approximate range of 20 to 20,000 Hertz (Hz), or cycles per second. Frequencies below this range are referred to as infrasounds whereas those above it are called ultrasounds. The frequency of each note is an absolute value by which it is uniquely identified. It is a known fact that *A* is tuned to 440 Hz; however we need to take this fact with a pinch of salt. The sound with a frequency of 440 Hz is one thing; another is the name given to that sound. This sound has been assigned the name *A* by convention. In this respect, the note *A* is as arbitrary as the metre standard that defines the metric system and was determined using a similar process. In 1939, a conference held in London set the standard diapason *A*, also known as the reference tone, at 440 Hz. This value had not been previously standardised and varied between different times and places and even for different instrument makers! Nowadays, leading orchestras throughout the world still prefer to set their *A* at different frequencies, although always in the region of 440 Hz.

PRIME NUMBER TROUBLE

In the early years of the 20th century, the initial standard for a standard *A* tuning fork was 439 Hz. How did it come to be defined as 440 Hz? According to a hypothesis from a member of the British Standards Institute, "The BBC tuning-note is derived from an oscillator controlled by a piezo-electric crystal that vibrates with a frequency of one million Hz. This is reduced to a frequency of 1,000 Hz by electronic dividers; it is then multiplied eleven times and divided by twenty-five, so producing the required frequency of 440 Hz. As 439 Hz is a prime number a frequency of 439 Hz could not be broadcast by such means as this."

Intervals and relative tuning

Before tackling the idea of relative tuning, we must first familiarise ourselves with the concept of an 'interval'. As we have just seen, each note has its own frequency, which identifies it from another. However, the Pythagoreans did not define each note separately, but in terms of the ratios between them. Given any two notes, these are separated by a notional distance, which is referred to as an interval. Once again, it is a good idea to make the distinction between the two ways of approaching this concept. One of these is to think of intervals as the musical 'distance' between two notes. Each interval is named by the number of notes that must be passed in order

to go from one to another, and by the direction of the path. Hence to go from *C* to *F* it is necessary to pass through four notes: *C-D-E-F*. The interval *C-F* is referred to as 'a fourth'. Or rather it is said that *C* and *F* are two notes that are 'separated by a fourth'. An octave, an interval with which we are perhaps most familiar, follows the same criteria. To go from one *C* to the next *C* it is necessary to pass through eight notes: *C-D-E-F-G-A-B-C*. The intervals that we have considered until now are ascending. Descending intervals run from the highest note and work downwards: an interval *C-A* is a third if we think of it going downwards: *C-B-A*. (The classification of intervals is explained in more detail in Appendix I.)

The other way of thinking about intervals is numerically, proportionally comparing the frequencies of each note. Between two notes there is a relative tuning so that what matters is not the absolute pitch of each, but the numerical proportion between the frequencies of two different notes.

This makes it possible to compare two notes referring to the interval that separates them by means of the numerical ratio between their frequencies. For instance, if we play two notes at an interval of a fourth, the highest will have a frequency of 4/3 the lowest frequency. If two sounds are separated by a fifth, the ratio between

LINEAR VS EXPONENTIAL GROWTH

In naming an interval, the number of notes between the two intervals, including the start and end notes, is counted. This makes the result of the sum of intervals counterintuitive. How much do a second and a third make? Do they make a fifth? A few simple calculations are all we need to show this is not the case. Assume a *C* as the start of the sum. Adding a second, we go from *C* to *D*. Adding a third, we go from *D* to *F*. Hence the sum is not a fifth but a fourth.

The sum of intervals behaves according to a certain linearity. If we number the keys of a piano, starting with 1 for *A0* all the way to 88 for *C8* we see that the *A* keys have the numbers 1, 8, 15, 22, 29... That is to say, when going from one *A* to another, there is a fixed increment of seven keys. However, when we take into account not the keys but the frequencies corresponding to these notes, we see that the growth is not linear but exponential. As such *A0* on the piano is tuned to 27.5 Hz. Passing to the next *A*, the frequency does not increase by a fixed value but is multiplied by 2, such that the next *A* is tuned to 55 Hz, the following to 110 Hz, and so forth.

the frequencies will be 3/2. For instance, from an *A* at 440 Hz, the *E* that is a fifth above is tuned to 660 Hz.

The ratio between the lengths of two strings is the inverse of the ratio of the frequencies of the strings. For example, two strings give notes a fifth apart, or 3/2, if their lengths have the ratio 2/3. From now on, we shall ignore all references to the lengths of strings and use frequency ratios in all cases.

Thus two notes with frequencies of 440 Hz and 880 Hz are said to form a perfect octave, and their tuning corresponds to the official *A*. Two notes with frequencies that are 442 Hz and 884 Hz are also perfectly tuned to form an octave, although

THE CURSE OF PERFECT PITCH

Perfect pitch is a skill that allows those who possess it to identify the name of the notes they hear without needing any other reference. Someone could play any key on the piano and a person with perfect pitch would recognise the note. There is no direct relationship between perfect pitch and potential musical talent. In fact, it is often the case that musicians are prejudiced by having perfect pitch. For example, in choir music it is highly common to shift scores to adapt them to the pitch that best suits the choir. By means of example, this is achieved by starting to sing in a semitone below what is written but reading the same score. Hence the notes that are sung do not correspond to those which are read, something that is often a source of conflict for those with perfect pitch, since it is hard for them to process the difference between what they read and what they hear.

their tuning does not correspond to the official A of 440 Hz. Finally, two notes with the frequencies 443 Hz and 887 Hz are not tuned to form a perfect octave, although the ear, can perfectly recognise their ratio as an 'untuned octave'.

The proportional link between the frequencies of two notes allows us to calculate another located at the distance of the desired interval based on a known sound, multiplying the first by the corresponding factor. If we know a frequency F_1, the ratio allows us to calculate F_2. For example if the notes were a fourth apart the calculation would be the following:

$$F_2 = F_1 \cdot \left(\frac{4}{3}\right).$$

This calculation can be 'chained' repeatedly, multiplying the corresponding factors. For example, if F_3 is the upper major third (which has a frequency ratio of 5/4) of F_2, it is possible to calculate the ratio between F_3 and F_1 by making substitutions like this:

$$F_3 = F_2 \cdot \left(\frac{5}{4}\right);$$

$$F_3 = \left[F_1 \cdot \left(\frac{4}{3}\right)\right] \cdot \left(\frac{5}{4}\right);$$

$$F_3 = F_1 \cdot \left[\left(\frac{4}{3}\right) \cdot \left(\frac{5}{4}\right)\right];$$

$$F_3 = F_1 \cdot \left(\frac{5}{3}\right).$$

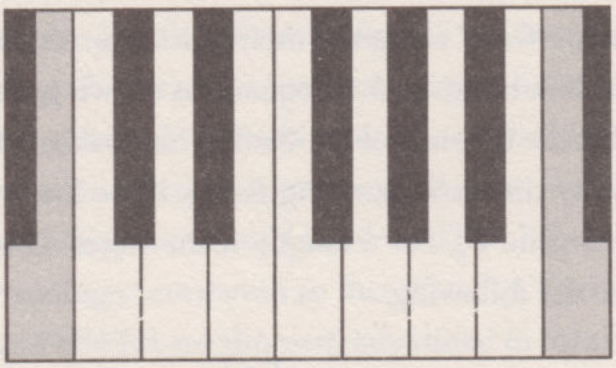
The calculations can also be made in the other direction, dividing instead of multiplying. For example, the frequency F_4, which is a fifth below F_1, is calculated as follows:

$$F_4 = \frac{F_1}{(3/2)}.$$

Both ways of thinking about intervals (that is to say their musical and numerical relationships) are intimately related. From now on we shall make use of whichever is most appropriate.

Tuning a piano

Now let's consider the task of tuning twelve notes of an octave on the piano.



C D E F G A B

To do so, we will make use of a procedure known as 'octave cancellation'. Once a *D* is tuned, it is transferred to all the other *Ds* on the piano, multiplying or dividing its frequency by 2; likewise for the other notes.

The initial value of *C* is normalised to 1. From this basis, a value between 1 (the *C*) and 2 (the following *C*), is assigned to each of the notes and corresponds to the proportional frequency of the note with respect to $C = 1$. We can thus determine the values of all the other notes. All the calculations can be replicated based on any other initial value (for example based on the *A* at 440 Hz).

The twelve notes imply that twelve steps are required to go from one *C* to the next. Each of these steps is known as a 'semitone'. We shall tackle the problem by first appealing to the discoveries made by the Pythagoreans, following the method they used for tuning the musical instruments in their time.

The Pythagorean scale

The Pythagoreans organised their scales based on simple numerical ratios between the different sounds. Thus the Pythagorean scale is structured around two intervals: the octave, with a ratio of 2/1 between the frequencies of the notes, and the fifth, with a frequency ratio of 3/2. The Pythagoreans obtained the different sounds of the scale by chaining fifths together, then using 'octave cancellation' to position these notes in the range they were seeking.

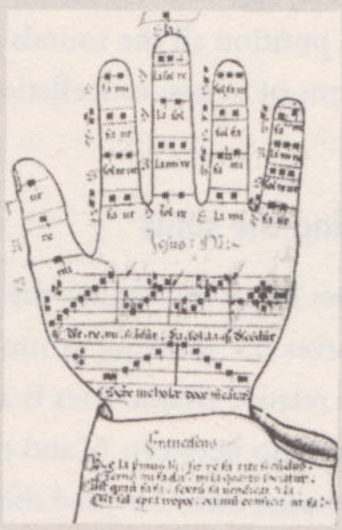
Let us start with *C* as an example. First of all, we calculate the ratio of the first ascending fifth in order to obtain *G*. Forming a chain gives us a *D*, then an *A*, an *E* and finally a *B*. Next, taking the fifth descending from the initial *C* gives us *F*. This gives us the seven sounds of the scale:

NAMING THE NOTES

The Greeks named the notes according to the first letters of the Ionic alphabet, assigning different letters for the same sound altered by half the tone or doubly raised. For instance, if *F* was alpha, beta was *F sharp*, and gamma, *F double sharp*. For these, the scale was arranged in descending order, in contrast to today's. The Romans also used the first letters of the alphabet to name the sounds of their scale. In the 5th century, the scholar Boethius wrote a five volume treatise on musical theory based on a fifteen-note scale spanning two octaves. Boethius designated each of the notes with a different letter, ignoring the cyclical concept of octaves. The next step in the history of naming the notes was to embrace this cyclical concept and use the same letter to designate equal notes in different octaves. This gave rise to the English and German naming convention designating the seven notes of the first octave using the upper case letters from *A* through to *G*; the following octave using the lower case letters from *a* through to *g*; and the third using double lower case letters (i.e. *aa*, *bb*, *cc*, *dd*, *ee*, *ff*, *gg*). As such, seven of the twelve sounds, those which correspond to the white keys of a piano, had their own name. The other five sounds (the black keys) were subsequently named following the invention of the concept of sharp, natural and flat notes. They did not have their own names, since they were derived from the seven basic notes. In the 16th century, the Tuscan monk, Guido d'Arezzo (c. 995–c.1050) dedicated a large part of his life to musical studies and the development of mnemonic rules for interpreters. Perhaps the most famous of these is the "Guidonian Hand" which ordered notes by their alphabetic notations, covering the palm of the hand. Guido d'Arezzo also renamed the notes, assigning each sound the first syllables of the verses of a hymn to Saint John which was well known at the time:

Ut queant laxis, **re**sonare fibris,
Mira gestorum, **fa**muli tuorum,
Solve polluti, **La**bii reatum,
Sancte **I**ohannes.

Hence, changing *ut* to *do*, resulted in the names of the seven notes of the scale used by the Italians, French and Spanish, among others. In English, the notation is used for the tonic sol-fa: do, re, mi, fa, so, la, ti. Do can represent any note on the piano, but is alwas the tonic (bottom note) of the scale.



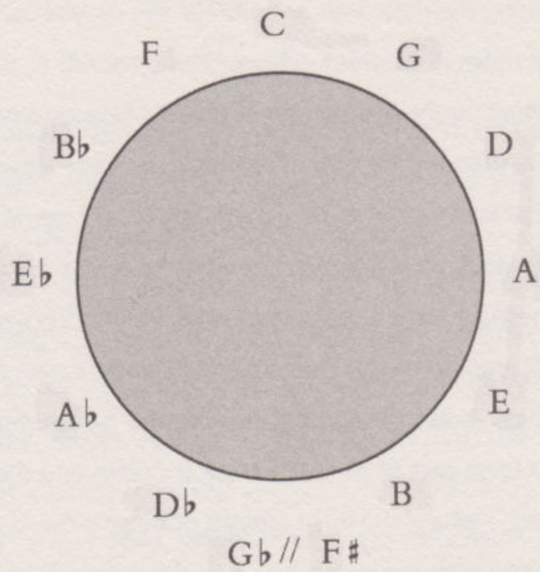
The Guidonian hand taken from a medieval manuscript.

$$F \leftarrow C \rightarrow G \rightarrow D \rightarrow A \rightarrow E \rightarrow B.$$

If we continue chaining fifths together, it is possible to obtain the eleven sounds that are referred to as the 'chromatic scale', made up of what is referred to as the 'circle of fifths'.

$$G\flat \leftarrow D\flat \leftarrow A\flat \leftarrow E\flat \leftarrow B\flat \leftarrow F \leftarrow C \rightarrow G \rightarrow D \rightarrow A \rightarrow E \rightarrow B \rightarrow F\sharp,$$

where the symbols *flat* (♭) and *sharp* (♯) designate decrements and increments of one semitone.



Having obtained the twelve notes by means of successive chains of fifths, you can position all the sounds on the same scale within the range of a single octave by means of octave cancellation.

Doing the sums

Now let us determine the tuning of each note by chains of fifths and 'cancelling' octaves (i.e. dividing or multiplying by two), such that, as we can recall, the value of the relative frequencies is always between 1 (the ratio between C and itself) and 2 (the ratio between C and the C of the next scale).

First of all, we determine G, which is a fifth of C:

$$G = \frac{3}{2}.$$

Next comes D, a fifth of G (multiplied by 3/2), but cancelling by an octave (multiplying by 1/2):

$$D = G \cdot \frac{3}{2} \cdot \frac{1}{2};$$

$$D = \frac{3}{2} \cdot \frac{3}{2} \cdot \frac{1}{2};$$

$$D = \frac{9}{8}.$$

The distance from C to D is referred to as a 'tone' and as one might expect, that is equivalent to two semitones.

Next comes A, a fifth of D

$$A = D \cdot \frac{3}{2};$$

$$A = \frac{9}{8} \cdot \frac{3}{2};$$

$$A = \frac{27}{16}.$$

Then E, a fifth of A, but 'cancelling' by an octave:

$$E = A \cdot \frac{3}{2} \cdot \frac{1}{2};$$

$$E = \frac{27}{16} \cdot \frac{3}{2} \cdot \frac{1}{2};$$

$$E = \frac{81}{64}.$$

The scale is completed with B, a fifth of E, and F, a fifth below C, and raised by an octave (multiplying by 2).

To summarise (with *C* having a value normalised to 1)

Notes	C	D	E	F	G	A	B	C
Frequency ratio	1	9/8	81/64	4/3	3/2	27/16	243/128	2

The process can be continued to determine the tunings of the black keys (flats or sharps), descending by fifths from *F*.

Note	D♭	E♭	G♭	A♭	B♭
Frequency ratio	256/243	32/27	1024/729	128/81	16/9

The Pythagorean comma

Ascending by a fifth from *B*, we reach *F*♯, which must be the same sound as *G*♭reaching the other end after having made the corresponding octave cancellations. However, these sounds are not exactly the same: the difference between *F*♯ and *G*♭ is known as the 'Pythagorean comma'. Similarly, after carrying out the corresponding octave cancellations, the limit sounds *F*♯-*D*♭ are not separated by a perfect fifth but differ by a Pythagorean comma. The slightly smaller fifth is referred to as the 'wolf fifth'.

Preparing the circle of fifths involves chaining twelve fifths together, reaching a note that is 'almost' the same as the one at the start, only with a difference of seven octaves.

This 'almost' is the Pythagorean comma. Its value can be calculated (let us call it PC) based on a frequency *f* and comparing the chain of twelve fifths starting from *f* with the chain of seven octaves:

$$\mathrm{PC} = \frac{f \cdot \left(\frac{3}{2}\right)^{12}}{f \cdot 2^7} = 1.013643265.$$

Therefore the difference is slightly more than 1% of an octave, or in equivalent terms, almost a quarter semitone. This difference is a result of the fact that the calculation of the fraction that defines the fifth is incompatible with the octave, as can be shown easily. To do so, we must search for any two components, x and y, which allow us to 'marry' the two fractions:

$$\left(\frac{3}{2}\right)^x = 2^y \Rightarrow$$

$$\frac{3^x}{2^x} = 2^y \Rightarrow$$

$$3^x = 2^x \cdot 2^y \Rightarrow$$

$$3^x = 2^{x+y}$$

From the last expression we can deduce that it would be the same as finding a number that was both a power of 2 and 3. However, and given that both 2 and 3 are prime numbers, this would contradict a fundamental theorem of arithmetic, according to which all positive integers have just one representation as a product of prime numbers. The first full proof of this theorem, postulated by Euclid, was provided by Carl Friedrich Gauss. From this it follows that the intervals of fifths and octaves defined by the Pythagoreans never even out, or rather, that there is no chromatic scale that is not accompanied by the inevitable Pythagorean comma.

Other tunings

Both the human voice and string instruments without fixed positions (eg without frets) allow 'natural tuning', in other words tuning that results in greater consonance between its notes, or greater harmony. This is the same in both cases because both are able to modify the pitch of the sounds emitted very subtly, correcting the tuning in order to achieve maximum consonance. As we have seen, the Pythagorean scale is constructed based on a starting note from which the others are determined based on successive chains of 'pure' fifths. However, this entails a number of arithmetical drawbacks in the search for a good level of consonance. The first is derived from the incompatibility of octave and fifth intervals which gives rise to the aforementioned wolf fifth. The second arises as a consequence of a different sort of incompatibility, in this case of fifths and major thirds.

In the Pythagorean scale, the tuning of thirds is carried out by chaining together four intervals of a fifth, which through cancelling by octaves is numerically equivalent to a frequency ratio of 81:64. However there is another way to determine the tuning of a third by following the simple ratio 5/4, or 80:64. These are the truly 'pure' thirds.

From this we can conclude that the Pythagorean scale, derived from a start point of chaining fifths, does not maintain the purity of thirds. There are three of these thirds on the white keys of a piano: *C-E*, *F-A* and *G-B*. It could be said that the Pythagorean scale prefers good fifths at the cost of 'impure' thirds.

The diatonic scale

The search for a 'pure' natural tuning has lead to a new way of organising sounds and their ratios known as the 'diatonic scale'. Although similar to the Pythagorean scale, which calculates all its intervals based exclusively on chains of fifths, the diatonic scale has a more complex arrangement.

Starting with *C*, it uses the fifth intervals in order to calculate the two notes that are considered as the next most important on the scale: *F* and *G*. It then calculates *E*, *A* and *B* as pure thirds of *C*, *F* and *G*, respectively.

The scale is completed with *D* tuned as a fifth based on *G*:

F	←	C	→	G	→	D
↓		↓		↓		
A		E		B		

The intervals of the diatonic scale are 'purer', i.e. acoustically more stable. This is also apparent in the increased simplicity of the ratios between frequencies that describe the intervals from which they are composed. First and always starting from a *C* normalised to the value of 1, *F* and *G* are calculated at a perfect fifth from *C*: *F* is tuned to 4/3, and *G*, to 3/2. Based on the *C*, its third, *E* is calculated at a distance of 5/4.

The same calculation is carried out to find *A* as a third of *F*:

$$A = F \cdot \frac{5}{4} = \frac{4}{3} \cdot \frac{5}{4} = \frac{5}{3}.$$

And again to find B as a third of G:

$$B = G \cdot \frac{5}{4} = \frac{3}{2} \cdot \frac{5}{4} = \frac{15}{8}.$$

And finally, D is calculated as a perfect fifth of G, cancelling by an octave:

$$D = G \cdot \frac{3}{2} \cdot \frac{1}{2} = \frac{3}{2} \cdot \frac{3}{2} \cdot \frac{1}{2} = \frac{9}{8}.$$

Note	C	D	E	F	G	A	B	C
Frequency ratio	1	9/8	5/4	4/3	3/2	5/3	15/8	2

The sequence by which the intervals of the diatonic scale were defined follows the structure of 'tonal music'. The vast majority of music from recent centuries is written in tonal music, from the Baroque and classical period to rock and pop music and Western folk music.

In tonal music, the notes are hierarchically organised around a main note, referred to as the tonic, or central tone. Each note has a 'musical' function in the organisation, with the different notes creating a play of tensions that propels the development of the musical process. This functionality means that certain intervals (especially sharps and flats, i.e. the black keys) are better tuned in a different way depending on the context in which they are used. The following table shows one of these possible tunings.

Note	D♭	E♭	G♭	A♭	B♭
Frequency ratio	16/15	6/5	45/32	8/5	16/9

There is always a problem…

The diatonic scale does not escape the problems that always arise due to the incompatibility of the main intervals of an octave: a fifth and a third. Almost all fifths are

3/2 although the fifth *D-A* is slightly less: 40/27. And the difficulties increase upon trying to complete the scale with sharps and flats – the wolf fifth always appears.

There have been a number of attempts to solve the problem using different 'temperaments', that is to say systems that attempt to reconcile the difficulties in the construction of the scale, partially relinquishing the purity of the tuning of certain intervals so that others are somewhat more acceptable. According to the greater or lesser purity of each interval, this option defines the colour of each temperament.

However, while these scales and temperaments achieve a relatively acceptable equilibrium for the different intervals of which they are comprised, it is an equilibrium that is always centred on the tonic, or rather, the note from which all others were calculated.

As long as this tonic remains as the centre, this is not a problem. However, when we wish to change the tonal centre, the entire configuration of the scale changes.

Although the absolute tuning of each of the notes is retained, the change of the tonal centre modifies the relative equilibriums depending on the new tonal centre, which results in a change of 'colour'.

Performing a work composed with its tonal centre on *C* on an instrument with tuning based on *C* using the diatonic scale, the work will sound just as it was intended. However, let us now assume that we wish to perform the same work, but in a higher pitch, that is to say centred on *D*, without changing instrument and keeping the tuning centred on *C*. The result is that the work will not only sound higher but also out of tune.

We can check this by considering the interval *D-A*. On the diatonic scale, the interval is not tuned to the ratio 3/2, but 40/27. In the new performance, with *D* as the tonic, the interval *D-A* would shift to occupy the place previously occupied by the interval *C-G*, tuned according to the ratio 3/2.

Everyone is happy

Thus far it has not been possible to construct a scale that does not include intervals that are out of tune. An inevitable question then arises: is it possible to construct a temperament in which all the ratios between the notes are maintained, regardless of the tonal centre? The problem cannot be solved by means of the compensation of intervals, altering the tuning of notes to expand or reduce them. The solution consists of defining the octave as being composed of twelve

intervals that are equally spaced from the outset. These twelve intervals must be twelve equal semitones which, when grouped together, form an octave.

Vincenzo Galilei, the father of Galileo, had already proposed dividing the octave into twelve equal semitones as early as the 16th century. The ratio between the frequencies of its semitones was 18/17. Chaining together twelve of these intervals results in 'small' octaves and fifths, of 1.9855 and 1.4919, respectively.

Let us approach the issue as if it were a simple mathematical problem. Let x be the frequency ratio that must exist between two consecutive semitones, such that twelve intervals of x are equal to an octave. Algebraically speaking, we can say that this x must satisfy the following equality:

$$x^{12} = 2 \Rightarrow$$

$$x = \sqrt[12]{2}.$$

By definition, the value of 1.05946 makes it possible to obtain a 'perfect' octave. The Pythagorean comma is equally distributed across all the notes of the scale.

As we have already seen, all the scales and temperaments used in one historical period or another distribute the Pythagorean comma according to the intervals that are deemed to be most important. These are kept totally pure, while the least important are distorted. In the temperament with the interval 1.05946, referred to as the 'equal temperament', all the intervals are equally 'distorted'.

In this system, the tuning of each interval is calculated by chaining together the number of semitones required for each case. Consider, for example, an interval of a fifth; this is composed of seven semitones, meaning the tuning is as follows:

$$x^7 = (1.05946)^7 = 1.49830708.$$

Applying this simple rule gives a scale of twelve notes and the value of the intervals is provided in the following table:

Notes	Value of intervals	
C	$(1.05946)^0$	1
C♯	$(1.05946)^1$	1.05946309
D	$(1.05946)^2$	1.12246205
D♯	$(1.05946)^3$	1.18920712
E	$(1.05946)^4$	1.25992105
F	$(1.05946)^5$	1.33483985
F♯	$(1.05946)^6$	1.41421356
G	$(1.05946)^7$	1.49830708
G♯	$(1.05946)^8$	1.58740105
A	$(1.05946)^9$	1.68179283
B♭	$(1.05946)^{10}$	1.78179744
B	$(1.05946)^{11}$	1.88774863
C	$(1.05946)^{12}$	2

Equal temperament has been imposed throughout the world, above all for instruments with fixed tuning, and in addition to this, the human ear appears to tolerate it extremely well. While it is certain that some intervals are perhaps too large (and conversely, others excessively small), it has two big advantages: the first, of a practical nature, is that it is possible to make use of the existing keys; the second, of a musical nature, is that thanks to the fact that all the intervals are identical, the system maintains its 'colour', regardless of its tonal centre (although it should also be pointed out that not everyone sees this as an advantage; some see it as a loss of diversity).

It is important to bear in mind that what we have described so far holds for any fixed instrument. It is the case with the piano, whose notes keep their tuning throughout the course of a musical interpretation. However freely tuned instruments, such as the human voice, are able to alternate between diatonic tuning and adapt to equal temperament when required.

Cents

The cent is a logarithmic unit of measurement used to measure extremely small frequency intervals with absolute precision. It is given by dividing each semitone into 100 equal micro intervals. A change in one cent interval is too small to be perceived by the human ear.

Just as there are twelve semitones in an octave, the cent is a number c such that

$$(c^{100})^{12} = 2 \Rightarrow$$
$$c^{1200} = 2 \Rightarrow$$
$$c = \sqrt[1200]{2}.$$

Cents provide us with a new way of comparing the measurement of intervals for different temperaments. As a logarithmic measurement, they are chained together by means of addition (and not by multiplication, as in previous cases). As such, cents can be used to facilitate many calculations. Given an interval p (expressed in its proportional measurement), its measurement in cents will be:

$$c(p) = 1.200 \cdot \log_2 p.$$

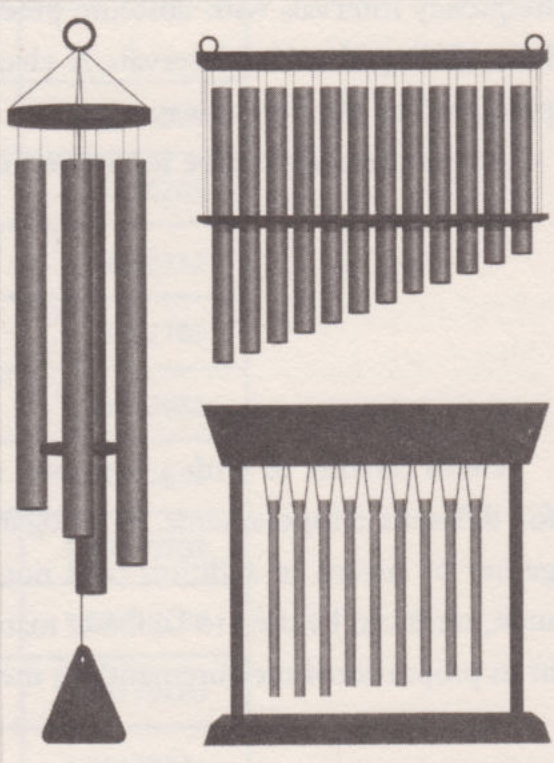
Thanks to this formula, it becomes possible to recalculate all the intervals and express them in cents for comparing the intervals in different temperaments.

		Do	Re	Mi	Fa	So	La	Ti	Do
Pythagorean scale	Proportional ratio	1	9/8	81/64	4/3	3/2	27/16	243/128	2
	Cents	—	203.91	407.82	498.04	701.95	905.86	1109.77	1200
Natural scale	Proportional ratio	1	9/8	5/4	4/3	3/2	5/3	15/8	2
	Cents	—	203.91	386.31	498.04	701.95	884.35	1088.26	1200
Equal temperament	Proportional ratio	1	1.1224	1.26	1.334	1.498	1.681	1.887	2
	Cents	—	200	400	500	700	900	1100	1200

HARMONIC CHIMES

Wind chimes are sets of small tubes, generally made of metal, of different lengths, which are fastened to a circular part. A ring hangs in the centre and hits the tubes when moved by the wind. The tuning of the tubes often corresponds to a pentatonic scale although they can be manufactured with collections of different sounds. Both the proportional length of the tubes and the point at which the hole is made on each of them to be hung must be precise. The scale begins with a tube L which forms its basic sound. The lengths of the other tubes, L_i, are calculated based on this one using the formula:

$$L_i = \frac{L}{(R_i)^{\frac{1}{2}}}.$$



Different types of chimes manufactured from metal tubes.

In comparison with pure fifths, the fifths of the equal temperament are a little small. For their part, the thirds of the equal temperament are half way between the other two, being larger than pure thirds but smaller than the Pythagorean ones.

Commensurability

Although the Pythagorean world was not familiar with fractions as we now know them, they used the equivalent concept of ratios between whole numbers. As we have seen, this special arithmetic allowed them to explain their discoveries with respect to the harmony of two strings when comparing their relative lengths: 2:1, 3:2, 4:3…

One of the strongest beliefs of the Pythagoreans and a fundamental aspect of their idea that numbers expressed the harmony of the Universe was that two given measurements were always commensurable, that is to say, they could be compared

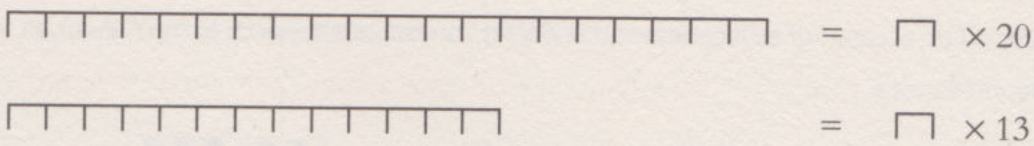
The frequency ratios R_i link each sound to the basic one. Similarly, the joint must be positioned at a height of 22.4% of the length of the pole. The following table shows some of the lengths for a seven tube chime.

Interval	R_i	L_i	Join
Basic	1	30	6.72
2nd	1.125	28.28	6.34
3rd	1.25	26.83	6.01
4th	1.34	25.98	5.82
5th	1.5	24.49	5.48
6th	1.67	23.24	5.20
7th	1.875	21.91	4.91
8th	2	21.21	4.75

Where necessary, it is also possible to calculate lower lengths, for example a descending fourth. In this case, the fraction of R_i is the inverse of the ascending fourth:

4th (descending)	0.75	34.64	7.76

using whole numbers. The concept of commensurability is directly related to what are now known as rational numbers. A rational number is what is commonly referred to as a fraction, division, relationship or ratio between two integers (where the second is non-zero). In modern terminology, a definition of Pythagorean commensurability would state that two given measurements *A* and *B* are commensurable if there is a third measurement *C* and two integers *p* and *q* such that *C* is *p* times *A* and *q* times *B*.



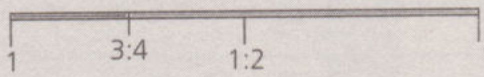
Or put another way, it can be established precisely using just two integers how many times greater (or smaller) two measurements *A* and *B* are with respect to the

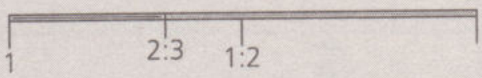
other. However, much to their consternation, the Pythagoreans were already aware of the existence of incommensurable numbers, i.e. those that could not be expressed as a ratio of integers and which are currently referred to by the somewhat unflattering term 'irrational'. The most famous irrational numbers are π and $\sqrt{2}$.

THE THREE MEANS

Pythagoras was influenced by his knowledge of the means (arithmetic, geometric and harmonic) and the mysticism of the natural numbers, especially the first four, known as 'tetrakis'.

As can be seen in the following diagram,





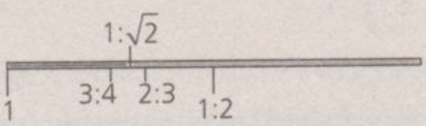
3:4 is the arithmetic mean of 1 and 1/2:

$$1-\frac{3}{4}=\frac{3}{4}-\frac{1}{2},$$

whereas 2:3 is the harmonic mean of 1 and 1/2:

$$\frac{1-\frac{2}{3}}{1}=\frac{\frac{2}{3}-\frac{1}{2}}{\frac{1}{2}}.$$

Pythagoras experimentally proved that strings with lengths in the ratios 1:2 and 2:3 (harmonic mean of 1 and 1/2) and 3:4 (arithmetic mean of 1 and 1/2) resulted in pleasant combinations of sounds and, as we have seen, constructed a scale based on these proportions. He called these intervals 'diapason', 'diapente' and 'diatessaron'. Today they are referred to as the octave, the fifth and the fourth. But what happened to the geometric mean? Was it rejected on account of its incommensurability? It corresponded exactly to the *F sharp* on the chromatic scale.



The latter arises when using Pythagoras' theorem to calculate the hypotenuse of a right-angle triangle whose catheti, or legs, have a value of 1. Ironically Pythagoras' master work undermined the rest of his strongly-held world view about the harmony of numbers.

As we have seen, Vincenzo Galilei's proposal to tune intervals using a whole number ratio of 18/17 meant it was not possible to obtain pure octaves. His choice of the ratio 18/17 is a good approximation, but we should ask whether there is another rational number equivalent to $\sqrt[12]{2}$, the measurement of the semitone of the equal temperament. That is to say, if there are two positive integers a and b such that

$$\frac{a}{b} = \sqrt[12]{2}.$$

The answer is that no such rational number exists. As a result, it is impossible to tune the semitone with an integer ratio a/b such that twelve semitones chained together result in a 'true' octave. If they did exist, we would have

$$\left(\frac{a}{b}\right)^{12} = 2 \Rightarrow$$
$$\left(\frac{a^6}{b^6}\right)^{2} = 2$$

and thus would have discovered two integers $a' = a^6$ and $b' = b^6$ such that $(a' / b')^2 = 2$, and thus that $\sqrt{2}$ would be a rational number, which we know to be false.

What would the Pythagoreans say if they saw – as you will at the end of this book – that it is the irrational numbers that finally solve the tuning problem?

Chapter 2

The Other Dimension: Time

I feel that rhythm is the primordial and perhaps essential part of music;
I think it most likely existed before melody and harmony,
and in fact I have a secret preference for this element.
Olivier Messiaen (1908-1992)

The Universe is ceaselessly changing. Time's passing reveals itself in that change: changes of position, of shape; chemical and physical changes… biological, meteorological, geological and astronomical processes, all these are situated in the passing of time. In this moving Universe, the actions of men and women do not escape time. Rhythm is the way these events are revealed in time, both in natural processes and those produced by human intentions.

The lunar phases and tides, the seasons, the days and nights, all dance to the rhythm of the heavenly bodies in their orbits. Fortunately, humanity has been able to surprise time with new steps, far removed from the rigid pulse of the clock. In short, we describe the succession and repetition of events with the concept of rhythm. In particular, musical rhythm is the frequency at which certain tones are being emitted.

Since prehistory, humanity has made attempts to preserve melodic expressions in a graphical manner. Neumes, primitive musical symbols, provide an idea of the phrasing and the intensity of the song, but do not transmit pitches or precise rhythms. It was necessary to know the melody, transmitted orally or by imitation, in order to be able to apply the musical sense provided by the neumes.

Rhythmical groupings: rhythm, beat, accents

When listening to good music, the listener may be struck by a need to accompany certain audible phrases with the movement of their foot, their hand or their head.

This first rhythmical grouping to be perceived is what is often referred to as 'beat'. The listener can intensify and attune their attention even more on the rhythmical content of each of these beats: an internal rhythm that is referred to as the 'breakdown of the beat'. There are two types of rhythm: 'binary', where the beat is subdivided into two, or 'ternary', where the beat is subdivided into three. If the beat is perceived to be subdivided into four, it is also referred to as binary. Larger groupings are often interpreted as being composed of smaller ones, for instance $5 = 3 + 2$ or $5 = 2 + 3$. It could be said that the rhythm is the heartbeat of a piece of the music.

From Greece to the first proportional notes

The first recorded historical example of musical notation comes from the Fertile Crescent in a tablet dated to around 2000 BC found in the Sumerian district of Nippur in modern-day Iraq. It describes a musical piece using the diatonic scale and composed in harmonies of thirds. Later on, the Greeks developed their own system of notation, which allowed them to represent the pitch and duration of a note, although not harmonies. The famous Seikilos epitaph, dated to between the 2nd century BC and the 1st century AD, contains the full musical record of a hymn. Above its words are a series of letters and symbols that modern researchers have been able to build into an approximate idea of the nature of the tune.

The different Greek systems of notation were left to be forgotten after the fall of Rome and it was not until the second half of the 9th century when, in order to record Gregorian chants, a new system arose in Europe, the 'neumic' system, derived from the syllabic structuring of Latin poetry. Neumes were primitive musical symbols, the shapes of which indicated the approximate note corresponding to the section of the word of the hymn. They provided a more or less complete idea of the phrasing and intensity of the song, despite being unable to imply exact rhythms or pitches. As such, some familiarity with the melody was required and this had to be communicated acoustically before a singer would have been able to apply the musical information expressed in the neumic notation. In order to overcome these limitations, the neumes were provided with a complex series of complementary annotations and were given different relative pitches, indicated using four parallel lines, the origin of the stave used today.

Towards the second half of the 13th century, a strong process of secularisation of the arts took place in Europe, which ran in parallel to a loss of prestige of the religious establishment. Until then, musical composition had been almost exclu-

A HYMN TO THE EPHEMERAL NATURE OF LIFE

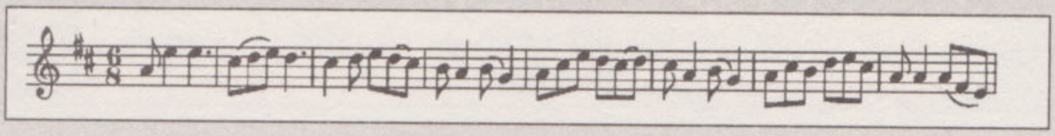
The Seikilos epitaph is engraved on a Greek tomb found in the district of Aidin in modern-day Turkey The full text of the epitaph reads: "I am a tombstone, an icon. Seikilos placed me here as an everlasting sign of deathless remembrance." The text is followed by a piece of music composed to the words of a hymn and for which a series of letters and symbols are provided. Transcribed into modern Greek writing, the piece has the following appearance:

C Z̄ Z̄ KIZ Ī K̄ I Ż İK O C̄ OΦ̄
Ὅ σον ζῇς, φαί νου, μη δὲν ὅλ ως σὺ λυ ποῦ·
C K Z İ K̇IK C̄ OΦ̄ C K O I Z K̄ C C̄ CXJ̄
πρὸς ὀλ ί γον ἐ στὶ τὸ ζῆν, τὸ τέ λος ὁ χρόνος ἀπαι τεῖ.

The text of the hymn can be translated as follows:

> "Shine while you live,
> do not suffer anything;
> life is very short,
> and time takes its toll".

In terms of the musical score, in modern notation it is as follows:

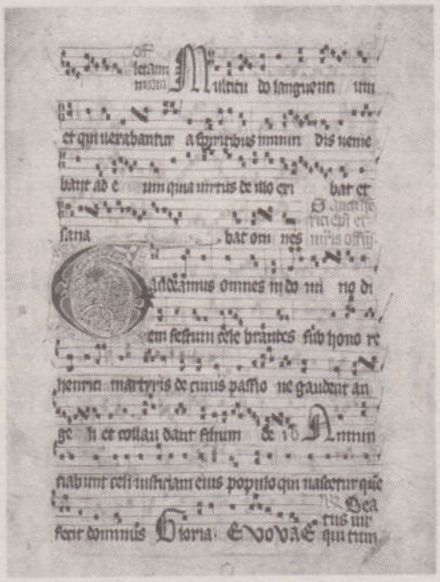


sively the concern of religious circles, and outside this environment, popular music had been characterised by a rich, polyphonic development that required a different notation.

Towards the end of the 13th century and the start of the 14th, a new, more efficient method of musical notation was developed, documented by Philippe de Vitry (1291–1361) in his treatise *Ars Nova*. His work, both theoretical and musical, was characterised by its emphasis on capturing rhythm, and had a graphical system for presenting this new polyphony, in which a number of voices were required to perform with great precision.

SYLLABLES AND MELISMAS

In Europe at the start of the 18th century, music was written down using neumes and a stave with four parallel lines, governed by a key that indicated the pitch.



A good example of the complexity obtained using neumic notation: a book of Finnish hymns from the Graduale Aboense*, between the 13th and 14th centuries.*

The hymns where each note corresponds to a single syllable are referred to as syllabic, whereas those in which the same syllable can be sung over a sequence of notes are referred to as melismatic. When the sequence of notes is ascending, the neumes that indicate them take on the appearance of pinched squares read from bottom to top; when the sequence is descending, they look like diamonds and should be read from left to right. For instance a neume showing a syllable sung with three notes has four variations.

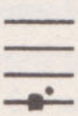
Perfectum/imperfectum

The *Ars Nova* was revolutionary in many respects, and captured what had already been outlined in various musical studies of the time. Until then, the prevalence of religious music had given preference to ternary breakdowns with the number three being associated with the Holy Trinity and, as such, perfection.

Philippe de Vitry's work laid the foundations of a notation that was able to respond to the rhythmical requirements of an increasingly sophisticated musical expression in which the ternary and binary coexisted, and clearly balanced the proportions between figures. The solution devised by the French musician and poet is

	*Scandicus*	Three ascending notes.
	*Climacus*	Three descending notes.
	*Torculus*	One note, followed by another ascending and a third descending.
	*Porrectus*	Ascending note, descending note, ascending note.

Neumic notation would later make use of symbols that look more familiar and clearly show the historical origins of modern notation.

	*Flat*	This mark has the same meaning as the current flat and was only indicated on the line for *B*.
	*Mora*	Similar to modern notation, a dot after a note indicates that it should be lengthened.

highly ingenious insofar as it includes both binary and ternary breakdowns using the same symbols or notes. The method is based on three proportional systems and creates a new note – the 'minima' – which was added to the existing collection made up of the 'longa', the 'breve', and the 'semibreve'. The three types of proportion between the notes are as follows:

– Mode *(modus)*: relationship between longa and breve.
– Tempo *(tempus)*: relationship between breve and semibreve.
– Prolation *(prolatio)*: relationship between semibreve and minima.

The first two proportions, mode and tempo, could be either:

– Ternary or perfect.
– Binary or imperfect.

For the third type of proportion, prolation, it was deemed:

– Minor if it was binary.
– Major if it was ternary.

The following table gives the proportions between the different notes and the equivalences in terms of the duration of these with respect to different proportions.

Proportion	Notes	Relationship between notes
Modus perfectum	Longa/breve	1 longa = 3 breves
Modus imperfectum	Longa/breve	1 longa = 2 breves
Tempus perfectum	Breve/semibreve	1 breve = 3 semibreves
Tempus imperfectum	Breve/semibreve	1 breve = 2 semibreves
Prolatio major	Semibreve/minima	1 semibreve = 3 minimas
Prolatio minor	Semibreve/minima	1 semibreve = 2 minimas

It should be noted that the binary or ternary property is not exclusive to the beat but is repeated in larger rhythmical groupings. For example, a bar may have two or three beats, and as such may also be called binary or ternary. Bars are nothing more than a convention designed to make it easier to understand and record rhythm. They are based on a simple principle – dividing a musical composition into equal units of times.

Bars are formed by accenting the first beat of a series of two or more such that they are grouped into a pattern. For instance, in the case of a binary beat, ONE

two, ONE two, whereas for a ternary beat, ONE two three, ONE two three. In modern notation, the time signature of a bar is indicated using a fraction x/y where x is the number of notes that fit in a bar, and y, the type of note (1 indicates a semibreve, 2 a minim, 4 a crotchet, etc.). We shall return to consider this point further on, together with other aspects of the bar and its notation.

For now though, let us return to Philippe de Vitry. With the combination of tempus and prolatio, together with the different types of beats, the *Ars Nova* achieved clearly differentiated and well-defined rhythms, which were symbolised in the following manner:

- A circle with a dot in the centre represented a ternary bar with a ternary breakdown, equivalent to the current 9/8 time signature.
- A circle without a dot represented a ternary bar with a binary breakdown, equivalent to the current 3/4 time signature.
- A capital C with a dot inside represented a binary bar with a ternary breakdown, equivalent to the current 6/8 time signature.
- A capital C without a dot represented a binary bar with binary a breakdown, equivalent to the current 2/4 time signature.

The following table summarises these four rhythms, in addition to the symbols used at the time and the equivalences between different notes. Note how in tempus perfectum, prolatio major, a breve note (the square) is equivalent to three semibreves (rhombuses or diamonds), each of which is in turn equivalent to three minims (a rhombus with a vertical bar).

Tempus perfectum	Prolatio major	9/8	⊙ ▪ = ♦ ♦ ♦ = ↓↓↓ ↓↓↓ ↓↓↓
Tempus perfectum	Prolatio minor	3/4	○ ▪ = ♦ ♦ ♦ = ↓↓ ↓↓ ↓↓
Tempus imperfectum	Prolatio major	6/8	Ͼ ▪ = ♦ ♦ = ↓↓↓ ↓↓↓
Tempus imperfectum	Prolatio minor	2/4	C ▪ = ♦ ♦ = ↓↓ ↓↓

PHENOMENA AND REPRESENTATION

In the same way that mathematics generates models that attempt to conceptualise reality, it is important to conceive of musical notation as a graphical representation of a phenomenon and not the other way round. When a musician comes into contact with the written material of a work they have never heard, they will only obtain an 'approximation' to the musical idea of the composer from the score. It is suffices to listen to different versions of the same work interpreted by different musicians in order to appreciate the diversity of perspectives. A similar thing happens with written texts when they are read or recited: a poem, for example, when read in an actor's voice has an inexplicable expressive weight and meaning, whose magic it is impossible to transmit on paper. A map of a country is only a two dimensional graphical representation of the land. It is not the country itself, but it can serve as an aid and guide for journeys. The score is able to communicate a technical aspect of the music, however its interpretation is in the hands of the musician, whose aesthetic decisions are what completes the message and gives it meaning.

Percussion: pure rhythm

Rhythm is clothed in melody, with its changes of pitch and intensities. In percussion, on the other hand, rhythm is laid bare. There are bursts of intensity, pitch and timbre that embellish them. But beyond these nuances, the beat is either there or it is not, there are no other alternatives. Thus rhythm is revealed more sharply, making it possible to appreciate its essence – ideal terrain for a mathematical expedition.

In percussion, cyclical sequences are characterised by the distribution of phrases. We will set out to note only these phrases, stripped of the prolongation to the notes caused by resonance. In this way it is possible to perceive the precise phrase and understand the sequence. We can distinguish three levels of rhythmical sensation according to the degree of intensity:

- The first level, the fastest phrase, corresponds to breakdowns of the beat. We will number them starting from the first beat as 1, 2, 3, etc., until reaching a new beat and restarting the count.
- On the second level are the beats themselves, corresponding to all the number ones in the sequence.
- On the third level are beats heard with greater intensity, known as accents.

A time signature of 9/8 expressed in the three levels we have just described would be as follows:

1st	1	2	3	1	2	3	1	2	3	1	2	3	1	2	3	1	2	3	…
2nd	1			1			1			1			1			1			…
3rd	1									1									…

Let us now take the second line, which only shows the beats, as our reference point. Filling the empty spaces with zeros gives us a clear representation of the sequence of beats. Each 1 represents a note; each 0 a pause. The result is pure rhythm.

1	0	0	1	0	0	1	0	0	1	0	0	1	0	0	1	0	0	…

The beats are the structure on which music is arranged; like the warp threads on which textiles are woven. We have been considering notes and pauses. Let us now set the 'quaver' as the measure of these beats, which we shall denote as ♪. Just as there are only notes and pauses, there are only quavers and quaver pauses (or rests). The latter will be represented by the symbol 𝄾. The most basic unit made up of a note and a pause (or a quaver and quaver pause) is referred to as a 'crotchet'and is represented using the ♩ symbol. Finally, let us introduce a new symbol into our notation, the 'dotted note', ♩., to indicate cases where note is held for longer, equivalent to the sequence ♪𝄾𝄾.

In summary, we have a binary system in which the values 1 and 0 are assigned to quavers and quaver pauses, respectively. In a 4/4 bar the equivalent of the symbols in our notation are written as:

♩	♩	♩	♩	=	♪	𝄾	♪	𝄾	♪	𝄾	♪	𝄾	=	1	0	1	0	1	0	1	0

Note how the sequence of note-pause-note-pause is repeated twice. A bar with two beats, each of which lasts for a dotted crotchet (6/8), is represented as:

♩.	♩.	=	♪	𝄾	𝄾	♪	𝄾	𝄾	=	1	0	0	1	0	0

Covering audible 'space'

The following chapter will analyse the structure of canons in greater depth. For the time being, we shall concern ourselves with the rhythmical aspect of such music, referred to as 'rhythmical canon'. Simultaneously playing various rhythmical cells represents an important challenge in terms of performance, both on an individual and collective level. One way of understanding this practice is performing the rhythmical canon.

Let us start with the rhythm ♩. ♩. ♩ = 3 + 3 + 2 = 10010010, cyclically performed by two musicians. The second begins their sequence after the first phrase of the first performer.

1	0	**0**	1	0	**0**	1	0	1	0	**0**	1	0	**0**	1	0	1	0	...
	1	**0**	0	1	**0**	0	1	0	1	**0**	0	1	**0**	0	1	0	1	...

In the previous sequence we can observe that at the points 3, 6, 11, etc. (marked in bold), neither rhythm is being performed. There are some rhythmical patterns that can be played 'in canon', such that:

a) two musicians do not begin to perform at the same time;

b) there is no point where no one is performing.

It is possible to compare this situation with the mathematical problem of the *tessellation of the plane*, that is to say the problem dealing with covering the entirety of a plane with regular geometric shapes. In our case, the plane we wish to cover is the audible one.

Naturally, a simple structure such as ♩. ♩. = 100100, meets the condition. However, the challenge increases as the basic rhythmical sequence grows longer. The following sequence of twelve notes:

1	0	0	0	0	1	0	0	0	0	1	1

completely covers the audible plane without any pauses when performed in a three-voice canon with entrances every three phrases. Let us examine the progression of the three voices:

1	0	0	0	0	1	0	0	0	0	1	1	1	0	0	0	0	1	0	0	0	0	1	1	...
				1	0	0	0	0	1	0	0	0	0	1	1	1	0	0	0	0	1	0	0	...
								1	0	0	0	0	1	0	0	0	0	1	1	1	0	0	0	...

PROPOSTA AND RISPOSTA

The word 'canon' was initially used to designate the rules for singers when performing voice pieces. However, from the 16th century, the word came to designate a specific type of composition in which a leader (also known as the *dux* or *proposta*) played the same melody that would then be repeated by a follower or followers (the *comes* or *risposta*). The melody of the follower could be rhythmically equivalent to that of the leader or, on the contrary, it could consist of transformations that increased or decreased its complexity. The children's song *Frère Jacques* is a well known example of a canon in which the followers repeat the same initial melody without any musical variation.

The transformations that can be used for the successive voices of a canon include: the number of voices; the 'waiting' time between the first and following voices, or even between the different voices where they are not always the same; the tempo of the melody being sung by the followers; whether the melody of the followers is an inversion of the initial one, a retrogression or even a combination of both, etc.

The canon was an extremely common compositional form in religious music and achieved its greatest expression in the work of late-Medieval composers such as Guillaume de Machaut and Renaissance figures such as Josquin des Près. However, the greatest exponent of the canon technique and, in general, of many other structurally more complex forms of music for which he gave examples considered as being perfectly 'canonical', was the great Johann Sebastian Bach.

First bars of the mass, L'Homme armé super voces musicales, *by Josquin des Près, the first movement of which consisted of a three-voice canon. The central voice is the slowest; the third sings at double the speed of the second, and the first at triple the speed. The lines join the first four notes of the composition for each of the three voices.*

From a mathematical perspective, it is interesting to consider whether there is a method that makes it possible to design sequences of this type. It must account for the following factors:

– Total number of phrases (p).
– Number of voices (v).
– Shift of the entry of the voices (s).

The following conditions must hold for the canon to have a solution:

– The number of phrases (p) must be divisible by v.
– The number of phrases, p , must be covered using v voices. As all the voices are the same, the structure must have a total of p/v 'ones'.
– The successive entries are shifted by a value equal to p/v, ensuring there are no positions in which ones are duplicated.

Let us consider an example with four phrases ($p = 4$) and two voices ($v = 2$). The structure must have an entry every $p/v = 4/2 = 2$ phrases. In this case it is possible to find all the alternatives and it is not hard to manually check which entries are correct. As such, the possible structures are:

1100
1001

As this sequence is repeated cyclically, it is easy to observe that both arrangements are the same. In the first case, the structure that only allows one movement of two phrases is:

1	1	0	0	1	1	0	0	1	1	0	0
		1	1	0	0	1	1	0	0	1	1

And for the second:

1	0	0	1	1	0	0	1	1	0	0	1
		1	0	0	1	1	0	0	1	1	0

It can be seen that once the canon is running, the performance is the same in both cases. Let us now consider an example of a structure with 12 phrases distributed in groups of equal duration. If we aim to cover these 12 phrases with 3 voices we must add,

$$p = 12$$

$$v = 3$$

$$\frac{p}{v} = \frac{12}{3} = 4,$$

it is necessary to arrange 3 groups of 4.

Starting from the following structure,

0000 0000 0000,

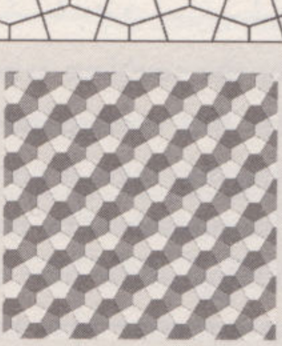
each of the four positions of the groups is occupied with a 1, such that they are 'all' occupied with *ones*, "just once":

1000 0100 0011.

To avoid the duplication of notes (columns of *ones*) a shift of p/v is required, in this case 4.

TILING THE UNIVERSE

A tessellation is a regular pattern of shapes covering a plane. The simplest and most common example involves paving stones and tiles. The tessellation must satisfy two requirements: no gaps are left uncovered and the shapes do not overlap. The square and the regular hexagon are two examples of simple geometric shapes that cover a plane. However tessellation also permits designs made up of irregular shapes, such as the one that adorns a large part of the paving stones on the streets of Cairo and which appears represented in the images below with a 3-D effect.



Bars, metres and breakdowns

The accent and the bar

A musical passage alternates strong beats with other weaker ones. In musical notation, the passage is structured by the strong beats: a bar begins with each strong beat which lasts until the next strong beat. The bar is therefore every rhythmical fragment between two strong beats. If the bars of a musical passage have the same duration and properties, the work is said to be regular; mathematically speaking, we say it is constant. Thanks to this regularity, we need only define these constant properties at the start of the work.

Types of bar

If the beats of the bar have a binary breakdown, the bar is said to be in simple time; if the breakdown is ternary, it is said to be in compound time. The content of the bar is indicated by means of a fraction, the concept of which changes depending on whether it is simple or compound. The numerator indicates the number of beats, and the denominator corresponds to a conventional note that measures the units of the beat. For example, a semibreve is equivalent to one unit of beat; a minim to two; a crochet to four. In a bar in simple time, the denominator indicates the note that measures the beat, while in a bar in compound time, the denominator is the note corresponding to the breakdown of the beat. The most common units of beat are the crotchet (a note and a pause per beat), the dotted crotchet (a note and two pauses per beat)

Let us consider some illustrative examples. A simple bar, with two crotchets per bar, 2/♩, is indicated by the fraction 2/4.

The 2 corresponds to the number of beats and the 4 indicates that the beat is a crotchet. A bar in compound time with two beats per bar will have two dotted crotchet notes: 2/♩.

But the problem arises of how to transform this into a fraction. There is no number that corresponds to the dotted crotchet (nor to any other dotted note), meaning that there is no way to express the duration of the beat on the denominator. This is resolved by indicating the note of the breakdown of the beat instead of the number of the note of the beat. In our example, the breakdown is represented by quavers; since there are two dotted crotchets per bar, there will be a total of six quavers. Hence the fraction is 6/8.

The notation of pure rhythm allows us to clearly see the binary alternation of notes and pauses, *ones* and *zeros*:

2/4

<				<				<			
1	0	1	0	1	0	1	0	1	0	1	0

3/4

<						<					
1	0	1	0	1	0	1	0	1	0	1	0

4/4

<								<								<							
1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0

For ternary bars, each one will be followed by two zeros:

6/8

<						<					
1	0	0	1	0	0	1	0	0	1	0	0

9/8

<									<						<								
1	0	0	1	0	0	1	0	0	1	0	0	1	0	0	1	0	0	1	0	0	1	0	0

The most common bars, according to the numbers of beats are:

	2 beats	3 beats	4 beats
Single time	2/4	3/4	4/4
Compound time	6/8	9/8	12/8

ALL THE BARS IN THE WORLD

As we have seen, the bar is a structure with a determined duration in which sounds and silences are alternated. Artistic considerations aside, it is interesting to analyse the ways in which a bar can be completed using different rhythmical groups. What follows is an interesting combinatorial set that can be prepared by a music student – and exhaustive cycle through the combination of a given selection of rhythmical groups and silences. Starting with a 4/4 bar we arrange four crotchet rhythmical groups denoted as A, B, C and D. The procedure is as follows:

Arrange a first bar:

4/4 | A B C D |

Based on this combination of four elements, we carry out what is known as a 'permutation of values', which is composed of four cycles of permutations:

Step 1: Select the last element in the bar (in this case, group D), which will be the generator object for the first series of permutations. Now proceed as follows: compose the first bar by placing D between B and C:

4/4 | A B ***D*** C |.

Step 2: In the following step, this D is placed in the second position, shifting element B to the right. The new bar is thus:

Irregularities

The examples we have just considered reflect rhythms with regular bars and beats, or rather in which all bars have the same duration, as well as the beats. However, this is not always the case. For example, in African music – and the American music derived from it – irregular rhythmical patterns are commonplace. These irregularities have also cropped up in academic music.

There is a rich rhythmic pattern that spans various types of music from America and Africa. Using our terminology, we can say it is made up of bars with three irregular beats. This means that while all the bars have the same duration, their content has beats with different durations. The bar is basically made up of two ternary beats and a binary one; using our terminology, two dotted crotchets and a crotchet, as shown:

4/4 | A ***D*** B C | .

Step 3: Finally, the element D is placed in the first position, completing the first cycle of the permutation:

4/4 | ***D*** A B C | .

The bar which we have obtained becomes the new combination for the next permutation. The procedure is now repeated: the element in the last position (on this occasion C), is taken as the object to be moved and we repeat steps 1 to 3:

4/4 | D A B C | D A ***C*** B | D ***C*** A B | ***C*** D A B | .

After repeating the procedure with B and finally with A, we return to the initial ordering:

4/4 | C D A B | C D ***B*** A | C ***B*** D A | ***B*** C D A |
4/4 | B C D A | B C ***A*** D | B ***A*** C D | ***A*** B C D | .

This system for the permutation of values exhausts the combination of sequences for the chosen rhythmical groups.

<								<								<							
1	0	0	1	0	0	1	0	1	0	0	1	0	0	1	0	1	0	0	1	0	0	1	0

Expressed with its breakdowns, the bar is made up of eight quavers, that is to say a total content equal to a 4/4 bar. However, it represents a completely different rhythmical reality since the 4/4 bar has four binary beats. This new bar, on the other hand, has a mixture of binary and ternary beats.

Such irregularities are often expressed with a time signature that measures the number of breakdowns in each beat, separated by a + sign; in our example, 3 + 3 + 2.

Layered rhythms

Various cultures throughout the world make use of a percussion practice that is referred to as 'polyrhythm', in which various motifs with a range of phrases combine together in a total, complex and organised audible product. The musical results are extremely beautiful, since there is great variety, both in the rhythmical and instrumental parts. A living art, observed and analysed by mathematics.

Even if each of the different rhythmical layers is essentially different, it may be the case that the rhythm is the same, but interpreted out of phase, as in a rhythmical canon, or in its retrograde or inverse version. The result is always polyrhythmic. The *pajarillo* (from the Spanis *pajaro*, 'bird'), the *seis corrido* (literally 'continuous six') and the *seis derecho* ('straight six') are three polyrhythmic forms of the Joropo, music from the plains of Venezuela and Colombia, which is characterised by the range of rhythms that are simultaneously performed by each of the instruments typical to 'llanera' ('music of the plain'), the bandola, the harp, the cuatro and the maracas. One of the rhythmical properties of Joropo is the crossed way in which the 6/8 bars are played. The result is:

0	0	1	0	0	1	0	0	1	0	0	1
1	0	0	1	0	0	1	0	0	1	0	0

An extremely widespread example of Latin American music that is also present in Europe is the cross 'three against two', commonly written with a bar of 6/8 and one of 3/4 played simultaneously. In our ones and zeros notation we have:

1	0	1	0	1	0	1	0	1	0	1	0
1	0	0	1	0	0	1	0	0	1	0	0

Amalgams

For freer compositional requirements and also in the interest of recording works from the pop world, a rich source of rhythms, the 20th century has seen new ways of expressing rhythms. As such, combinations of sequences of bars with different durations have been used, such as groupings of seven crotchets, either using a combination of a 3/4 bar and a 4/4 one, or a sequence of 2/4, 3/4, 2/4. In a bar with five crotchet beats, these can be grouped as 2/4 and 3/4 or vice-versa.

The score from Igor Stravinsky's Concertina for String Quartet *in which it is possible to appreciate the enormous rhythmical variety of this great innovator in 20th-century music.*

Speed: the metronome

Once a rhythmic sequence has been written down, with its notes and pauses, we are presented with the challenge of reproducing it. For a performer who has never heard the piece of music, they will be able to reproduce the rhythms set out by the composer, although they will be lacking an essential component: speed. This aspect of the development of music through time is indicated at the start of the work and at any other point where there is a change in speed. The indication is a musical note accompanied by a number. In general, the note corresponds to one beat in the work and the number indicates the number of times this beat is repeated in a minute. Thus, the indication

♩ = 60

means that one minute contains 60 beats. In this case, all we need is a watch in order to calculate the speed of the work since the beats coincide with the seconds. For other speeds, we can make use of a metronome, an instrument that marks

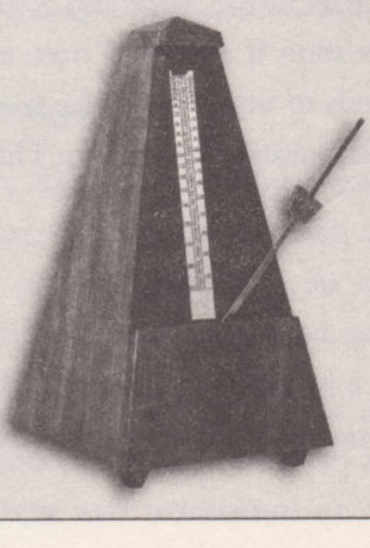
regular beats. In the case of the mechanical metronome, this is composed of a pendulum with a counterweight that can be adjusted up and down to alter the frequency of its oscillation. The greater the frequency, the more beats per minute. Musicians make use of metronomes to ensure regularity when playing an instrument, although it is also a reference tool that allows the composer to record the speed at which a piece is performed. The first person to establish the speed of performance for one of his compositions by this method was Ludwig van Beethoven.

Even if the metronome offers an objective measure of speed, as a guide for performance, it suffers from an excess of rigidity: when performing, a certain slippage of the strict and rigid beat is both common and natural. One unexpected although not uncommon use of the metronome is as a percussion instrument in certain works, such as the well-known Beatles song "Blackbird", on the *White Album*. The eminent Italian film score composer Ennio Morricone used distorted and slowed down recordings of a metronome on the track "Farewell to Cheyenne" on the soundtrack to *Once Upon a Time in the West*.

The most extreme example of the use of a metronome as an instrument has been by the Romanian György Ligeti, who, in his *Poème Symphonique for 100 Metronomes*

THE MUSICAL CLOCK

The mechanical metronome was invented by the German Dietrich Nikolaus Winkel in 1812, although the first patent belongs to his fellow countryman Johann Mälzel. Even if nowadays there are digital metronomes, the devices were originally pieces of clockwork. The most common has an internal clock mechanism and an inverted pendulum made up of a rod and counterweight that can be moved along its length. It had a double counterweight system,



one at each end of the centre of oscillation, the external one is adjustable and one inside is fixed. When the external weight is positioned closer to the centre of oscillation the period is shorter, i.e. faster. On the furthest end, the beat that is marked will be slower. For every oscillation, the internal mechanism produces a click. On some metronomes, it is possible to adjust the instrument so that a special sound marks the accents every one, two or three beats. There are currently electronic metronomes which are often sold with a tuning fork that emits a 440 Hz *A*.

EINSTEIN'S DIFFICULTIES WITH MATHEMATICS

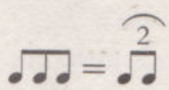
The physicist Albert Einstein, inventor of the theory of relativity, was also an enthusiastic amateur violin player, although perhaps he did not play the violin quite as well as he performed as a scientist. On one occasion, the scientist was practising a sonata in the company of an eminent pianist and fellow countryman, Artur Schnabel, and repeatedly became lost in a certain passage of the work, causing Schnabel to stop. The third time that this happened, Schnabel looked at him in frustration and blurted out: "For heaven's sake, Albert, can't you count?"

(1962), played one hundred of the devices at the same time. The piece ends when the last of the devices stops swinging.

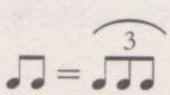
Isolated irregularities

On certain occasions it is necessary to make sporadic use of ternary beats in a work with a regular beat pattern (e.g. binary). With the elements that we have covered thus far, in order to indicate a change of this type we would need to allocate both a change of speed and a change of bar at the moment in question. In order to avoid this double reading, 'duplets', 'triplets', etc. have been designed:

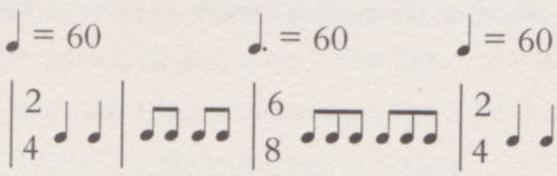
– Duplet: a group of two notes played in the time corresponding to three.



– Triplet: a group of three notes played in the time corresponding to two.



They are denoted by the use of an arc that joins the group of notes and a number corresponding to the new number of notes. Let us consider an example of a rhythmical phrase with a complex notation and a change in bar and metronome:



THE MATHEMATICS OF TIME SIGNATURES

It is interesting to compare the equivalences between time signatures and fractions and their operations. To what extent are operations that are valid as fractions also correct when working with time signatures?

– Fraction addition. For example, a 3/4 time signature has a duration of a dotted minim, equivalent to a minim (represented using the 𝅗𝅥 symbol) plus a crotchet:

𝅗𝅥. = 𝅗𝅥 + ♩

If the notes are replaced by their numerical equivalents, we have

$$3/4 = 1/2 + 1/4.$$

– Smplification of fractions. Simplifying a time signature gives a result that is comparable to a new time signature.

$$6/8 = 3/4.$$

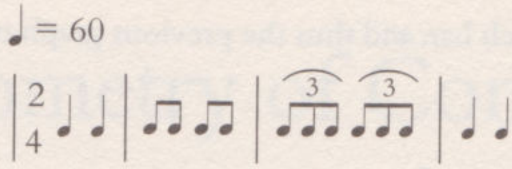
In this case, numerical equality does not correspond to musical equality. The duration of both bars is the same, six quavers (in the case of the 3/4 bar, three crotchets, with two quavers each). However the indication 6/8 corresponds to compound time, and 3/4 to simple time, an important distinction in music.

– Least common multiple.Of the various instances of polyrhythm, we are interested in the ones where two rhythms come into contact, one in compound time and the other in simple time in the same beat or bar. For instance, there are two crotchet beats in one bar, and in the other there are three crotchets in a crotchet triplet:

3
♩♩♩ = ♩♩

This causes a disjunction in the precise execution of each rhythm. The solution is mathematical and the process is the same as for calculating the least common multiple. In this example, we have LCM (2,3) = 6. This means that the bar should be thought of as divided into six equal parts: the crotchets are played on the divisions 1 and 4, and the crotchet triplet on 1, 3 and 5.

The same structure, now simplified by the use of triplets:

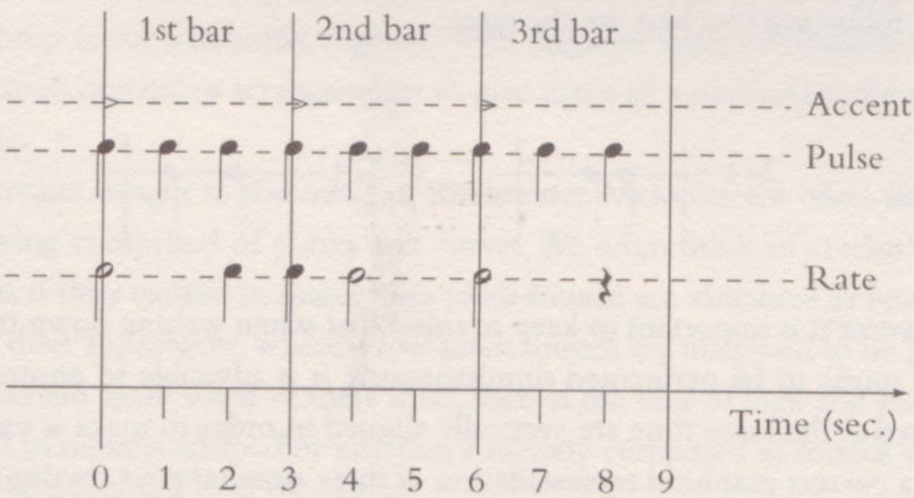


Modern notation

As a symbolic system, musical notation has achieved a significant degree of effectiveness over the centuries. In order to achieve its objective it has positioned variable elements (notes and pauses) and constant ones (tempo, key, bar) on a system of reference (the stave). Let us consider a full example.

The speed is constant: ♩ = 60.

The beat uses crotchets and one beat is accented for every two that are not, meaning we have a 3/4 bar. The following picture shows the sequence of beats and accents as if the score was a system of coordinates, with the time in seconds displayed on the *x* axis.



The accents occur regularly at intervals of three seconds.

The beats occur regularly at intervals of one second.

Reading all this information from the picture above is quite tiresome, and a rhythm score makes use of keys in order to simplify it. To do so, the constant values are indicated once at the start: beat and accent and hence the time signature, using the system of fractions we have already considered; in this case 3/4.

The speed is given using the metronome notation: ♩ = 60.

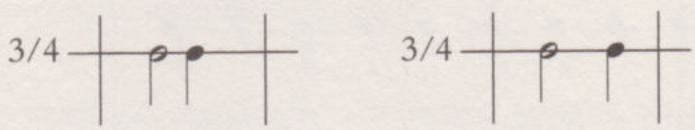
Sticking with the idea of representing the passage of time horizontally from left to right, the same way that books are read in the West, we can now add vertical lines to indicate the end of each bar, and thus the previous graph of the rhythm is simplified to:

With the position of the vertical lines and knowing that the beats occur regularly, the notes need no longer be placed with respect to a time scale.

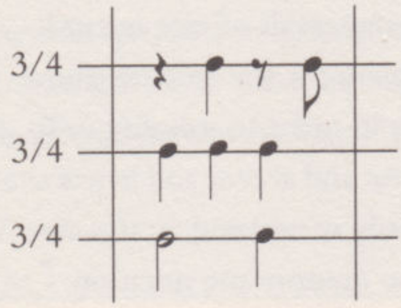
Without a scale

The horizontal line of a score is not a scale representation of time. This means that the duration of the notes or pauses does not necessarily correspond to the spacing between the notes, but to their relative content. The sound should be played with the duration determined by the note itself or the pause, not in line with the distance on the image.

The following two bars are the same:



However it is important to keep in mind that when writing down two or more lines of music to be performed simultaneously, it is advisable to ensure notes that are played at the same time are vertically aligned in order to make it easier to read. Hence a correct graphical representation of three simultaneous rhythmic phrases is indicated in the following manner:



Chapter 3

The Geometry of Composition

The vase gives form to emptiness, and music to silence.
Georges Braque

People accuse me of being a mathematician,
but I am not a mathematician, I am a geometer.
Arnold Schönberg

Nature organises shapes in a curious manner. In this universe of shapes, the gaze of mathematics uncovers the numerical and geometric patterns that characterise plants, animals, sounds and crystal structures. Spherical shapes, cyclic sequences and spiral orderings occur frequently, together with different types of symmetry. Artists feed on nature's capricious arrangements to give material a new order, the order of the aesthetic.

Music creates images in the mind of the listener. Melodies are often associated with a drawing comprised of points and curves. We often think of certain musical properties as if they existed in space: high pitch sounds are visualised as being high and thin in their appearance, whereas low pitch sounds are imagined to be low and thick. In a certain sense, some of these ideas, such as the idea of high and low notes, are reflected in musical scores. For instance, a melody composed of sounds with increasing pitch is referred to as 'ascending'.

This gives the score an additional role, accompanying the music with an image in the same way pictures do in an illustrated book. Many composers have taken these ideas into account when composing their works, and the history of music is filled with examples of pieces of music with fascinating designs that straddle music, writing and geometry. (In order to better appreciate the examples in this chapter, we suggest reviewing the fundamentals of modern musical notation by reading through the material in Appendix I.)

Pitch and rhythm: the musical plane

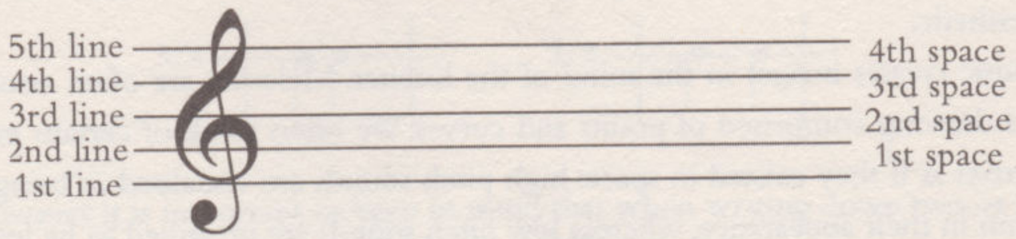
The elements of musical notation

The current system of musical notation is a product of a process of evolution aiming to give written expression to this most ethereal of art forms. This development has seen the incorporation and modification of a series of symbols and graphical elements and it is interesting to evaluate these from the perspective of mathematics and logic.

The stave

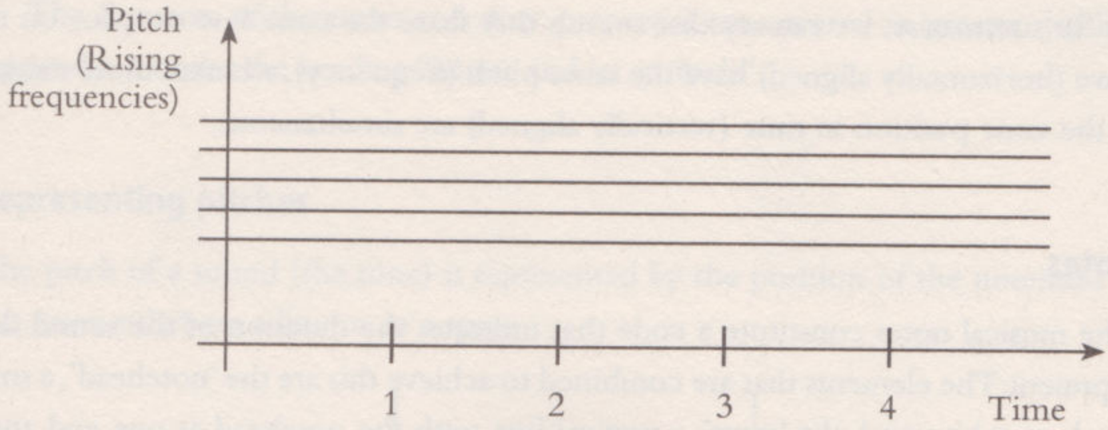
The graphical device used to represent music is the stave, which can be conceived of as the development of sounds and their pitches in time. If we compare it with a coordinate system, we have 'time' on the horizontal axis, and 'pitch' on the vertical one. This second variable is indicated by means of a series of parallel and equidistant straight lines, the number of which, in modern times, is fixed at five.

The 'musical distance' between two neighbouring lines (or between two neighbouring spaces) is an interval of a third, whereas between a line and its neighbouring spaces, the interval is a second. Hence we have five lines and four spaces, numbered from bottom to top:

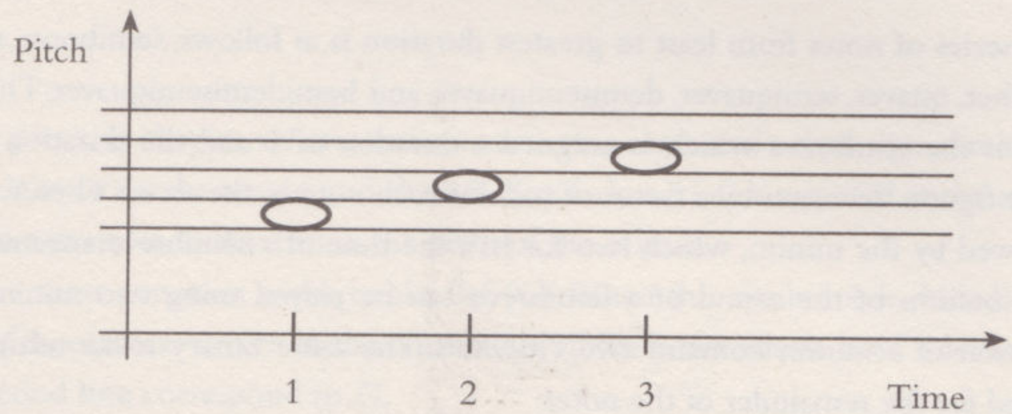


The distribution of lines and spaces is equivalent to the white keys on a piano and their number is directly proportional to the frequencies of the notes. Hence sounds with a greater frequency (the highest notes) are positioned higher up on the stave. Additional lines can be added to represent higher or lower notes. This obviously results in additional spaces. In fact, the first space above the fifth line and the first below the first one are additional spaces.

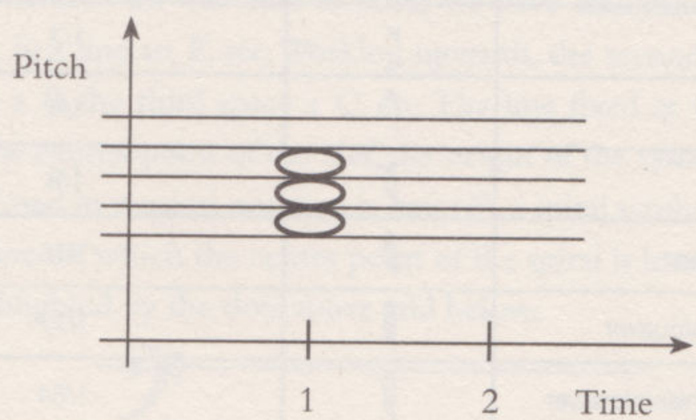
Returning to the idea of the score as a coordinate axis suspended in the 'musical plane', we can see that the ordinate axis represents the pitch.



The abscissa (items plotted on the x axis), on the other hand, represents notes and pauses in time. By means of example, three sounds with ascending pitch which are played at points 1, 2 and 3 are represented in the following way:



If these three sounds were simultaneously reproduced at point 1, they would be represented as follows:



To summarise, we can say that sounds that share the same line or space on the stave (horizontally aligned) have the same pitch (frequency), whereas those marked at the same position in time (vertically aligned) are simultaneous.

Notes

The musical notes constitute a code that indicates the duration of the sound they represent. The elements that are combined to achieve this are the 'notehead', a small black or white oval, the 'stem', a vertical line with the notehead at one end and a small curve referred to as the 'flag' at the other (where appropriate).

The series of notes from least to greatest duration is as follows: semibreve, minim, crotchet, quaver, semiquaver, demisemiquaver and hemidemisemiquaver. The basic note is the semibreve which is assigned a duration of 1 and the duration of the other figures decreases by a factor of two for each note in the series. Therefore, it is followed by the minim, which lasts for half the time of a semibreve, meaning that the duration of the sound of a semibreve can be played using two minims. The duration of a minim contains two crotchets. The same binary relationship is repeated for the remainder of the notes:

Name	Note	Duration with respect to the semibreve
Semibreve	𝅝	1
Minim	𝅗𝅥	1/2
Crotchet	𝅘𝅥	1/4
Quaver	𝅘𝅥𝅮	1/8
Semiquaver	𝅘𝅥𝅯	1/16
Demisemiquaver	𝅘𝅥𝅰	1/32
Hemidemisemiquaver	𝅘𝅥𝅱	1/64

The function of the notes and their properties are covered in greater depth in Appendix I, under the heading "Music and its symbols".

Representing pitches

The pitch of a sound (the tone) is represented by the position of the notehead on the stave, either on a line or in a space.

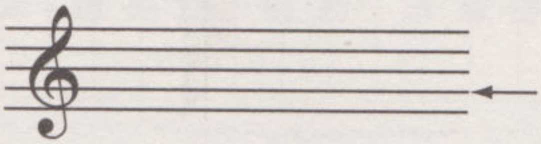
On a line In a space

Thus far, the information is only partial. To determine the absolute relative pitch between sounds it is necessary to make use of a clef, which reveals this secret.

Clefs

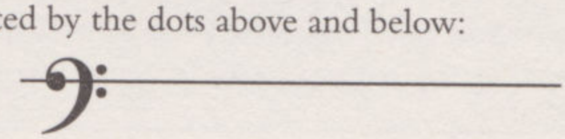
In the previous chapter we saw that a metronome mark and a time signature shown at the start to fix the tempo and rhythm. Similarly, a 'clef' is required at the start of the stave in order to determine the pitch of the sounds.

The most common is the treble clef (*G*). When this symbol is included at the start of the stave, as shown in the illustration, all the notes whose head is located on the second line correspond to *G*.

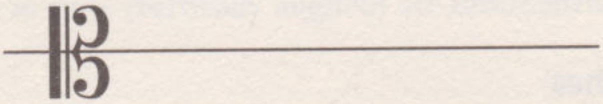


Working downwards on the scale of notes we have established, the first space will be an *F*, the first line an *E*, etc. Working upwards, the second space will be an *A*; the third line a *B*; the third space a *C*, etc. The line fixed as *G* is the one that passes through the central point of the clef, the origin of the symbol.

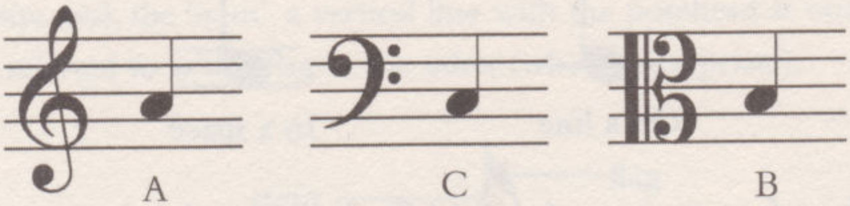
Another clef used in musical notation is bass (*F*), a spiral symbol that assigns the sound *F* to the line on which the centre point of the spiral is located. The position of the line is highlighted by the dots above and below:



The alto clef (C) is expressed using a symmetrical symbol that indicates that the line situated on its axis of symmetry is a C:

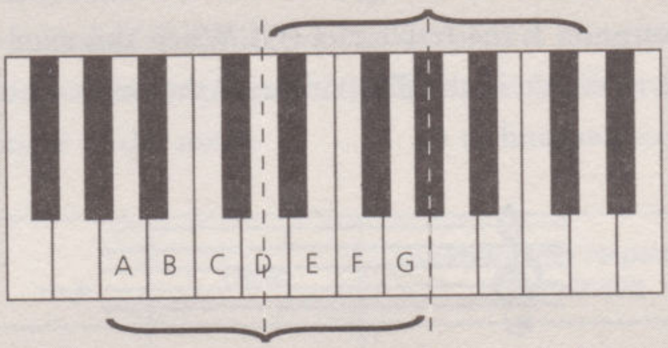


The pitch of the lines and spaces is altered depending on the position of the clefs. A symbol in the same position may correspond to various notes depending on the clef used.



SYMMETRY ON THE KEYBOARD

The keys of the piano have two axes of visual symmetry: one on the white key *D* and the other on the black key *G sharp*. Coincidentally, the central note of the European naming convention (ABCDEFG) is *D*, with the other six sounds distributed on either side of this axis of symmetry.



Let us consider how tones and semitones are distributed in scales. A major scale is one in which the series of seven sounds are arranged according to the following distribution of tones (T) and semitones (sT):

T-T-sT-T-T-T-sT.

The major scale which only makes use of white keys starts on *C*:

C, D, E, F, G, A, B, C.

We now see the same sound (middle C) also represented in three different keys:

Key of G, on line 2 — Key of F, on line 4 — Key of C, on line 3

The diagram on the previous page shows a note in the same position on the stave but whose value is altered by different clefs. On these lines, a single sound is represented in three different clefs.

The term 'natural minor scale' refers to the series of seven separate sounds defined by the following arrangement of tones and semitones:

T-sT-T-T-sT-T-T.

The minor scale which only makes use of white keys starts on *A*:

A, B, C, D, E, F, G.

This is precisely the case with symmetry whose axis is on the key *D* which we saw on the keyboard. It does not require much effort to note that the tones and semitones are symmetrically distributed:

A T *B* sT *C* T *D* T *E* sT *F* T *G* T *A*

We can find the other axis of symmetry between *G* and the next *A*. It is also clear that for this axis there is a symmetry of intervals between keys. What at first sight appears in the distribution of black and white keys on the keyboard has its equivalent with respect to the same axes of symmetry in the distribution of tones and semitones. As our current audible range is a chain of twelve equal semitones, taking any note as the central tone, it will be possible to find a collection of notes for which tone and semitone intervals are arranged symmetrically on the keyboard.

Semitone variations

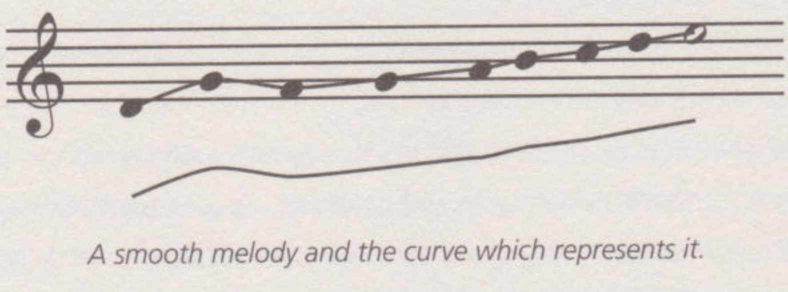
On some occasions it is necessary to adjust the pitch of a given note. There are two signs to increase or decrease the pitch of a note by one semitone: the ♯ sign (sharp) raises it and the ♭ sign (flat) lowers it. There is a third symbol that cancels the sharp or flat effect: the natural symbol (♮).

♯	♭	♮
Sharp	Flat	Natural

These symbols are placed on the line or space that is to be modified and affect the notes to the right of them for the remainder of the bar. When they are placed at the start of a work (between the clef and the time signature), these symbols indicate that, for the duration of the work (or at least until a new change), that all the pitches on that line or space should be adjusted.

The melodic curve

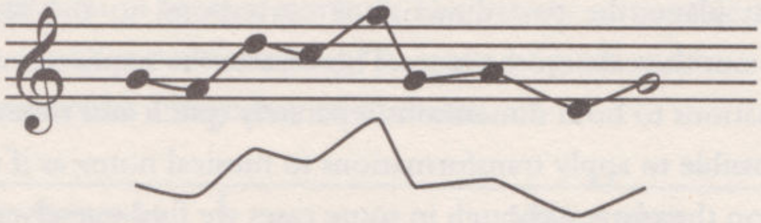
Without needing to know how to read or write music, listening to a melody often causes the listener to imagine a line with curves and straight lines, at times ascending and at others descending. It is highly probable that this curve is imagined moving from left to right, as if it were a written sentence. We think of certain melodies as having smooth curves, without large jumps whereas in contrast, others have marked changes in pitch. It is interesting to note the similarity between these images and the drawing that can be made of the melody with notes on a stave. Let us now take a score and as in a 'dot-to-dot' game, join the noteheads with a continuous line.



A smooth melody and the curve which represents it.

If we listened to the reasonably smooth line of the above example, we would

hear that it was a melody without jumps. Similarly, a melody with big jumps will result in a curve with significant changes in pitch, such as the one below:



A melody with large changes in pitch.

Geometrical-musical transformations

According to Gestalt psychology (a term that does not permit a single definition but can apply to concepts such as 'shape', 'structure', and 'layout'), the mind is able to select and group the parts of a whole and order this into a shape that distinguishes it from the remainder. This same process occurs in time thanks to our memories, which allow us to comprehend movement in a strip of animated drawings or appreciate the construction of a musical work. Gestalt has been used to study the processes of perception and has established certain principles which characterise them. According to the principle of closure, our mind tends to spontaneously complete incomplete shapes. Thus images with partial information, such as impressionist landscapes created using on thousands of small loose brush strokes, magically transform into a credible and continuous whole when seen from a certain distance. The same happens with still images in films, their continuous movement really just being an illusion. The laws of Gestalt can also be applied to music, allowing a listener to identify patterns and similarities among audible events which take place in time in a similar way to a viewer watching the images of a film.

Numerous geometrical concepts have been employed deliberately by composers as a compositional tool. In some cases, the geometric-musical aspect is visually present in the score, whereas in others it is found in the sound. Certain compositions have formal structures with geometric peculiarities, such as the canon. Its repeating form sets inflexible criteria for the melodies, turning it into a double challenge for the composer to create music while respecting strict mathematical criteria. Other works reflect the deliberate use of geometric transformation as a compositional device.

In this section we shall provide a comparative description of various geometrical transformations and some specific combinations of sounds. It is important to keep in mind that there is a fundamental difference that conditions the comparisons. On a flat plane, the two dimensions correspond to the same magnitude, whereas this is not the case on the score. This creates the requirement to apply musical transformations to both dimensions separately (pitch and time).

It is also possible to apply transformations to musical notes, as if they were geometric shapes on the plane, although in some cases we find ourselves confronted by an experiment that does not result in audible differences.

Similarly, it is important to keep in mind that transformations are carried out on the curve formed by the noteheads. Let us consider an example with a four note melody which, when joined in a straight line, give the following design:

After applying a transformation to the drawing…

…the corresponding stems and flags are returned to the noteheads:

Geometric-musical transformations provide composers with an additional device in their tool kit.

Isometric transformations

The term 'isometric' means that distances are preserved. There are three types of isometric transformations on the plane: translation, reflection and rotation, which have their correlation in musical symbols. The variety of possible transformations increases if we consider transformations in terms of pitch and time separately. The following table provides a summary of all these:

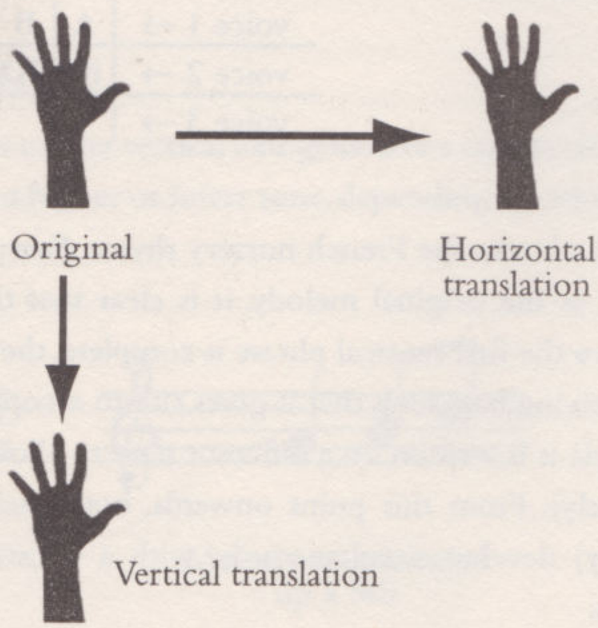
Geometric transformation	Musical result		
	Horizontal	Vertical	Horizontal + vertical
Translation	1. Repetition 2. Canon	Transposition	1. Ostinato 2. Canon – 2nd, 4th, etc.
Reflection	Inversion	Retrogression	
Rotation (180°, understood as a combination of two reflections)			Retrograde inversion

The number of possible alternatives is increased by the combination of some of these transformations:

Combination of transformations	Musical result
Vertical transposition + vertical reflection	Retrograde transposition
Vertical translation + horizontal reflection	Inverted transposition

Translations

The term translation refers to a geometrical transformation which, when applied, moves the shape in question in a given direction, without modifying its shape or causing any rotation. In our case, we need only consider horizontal and vertical translation as shown in the drawing on the right.



Horizontal translation: repetition and canon

In the case of the score, horizontal translation implies a translation in time and has two musical expressions:

Repetition: A melody, or a fragment of one, is performed a number of times consecutively.

$$O \rightarrow O \rightarrow O \rightarrow O \rightarrow O \rightarrow O \rightarrow O \rightarrow O \rightarrow O$$

In the simplest of cases, horizontal translation is just the repetition of a pattern, continuing the melodic line.

Original melody — Melody transposed horizontally

The canon: as we saw in the previous chapter, the canon is a musical structure in which a melody is simultaneously performed by various voices, each waiting for a short interval after the proceeding voice before starting.

voice 1 →	A	B	C	D	...
voice 2 →		A	B	C	...
voice 3 →			A	B	...

Let's take the French nursery rhyme *Frère Jacques*. If we take the first four quavers as the original melody, it is clear that this melody is repeated (translation). Once the first musical phrase is complete, the melody continues to develop in the following bars such that it gives rise to a copy of the original melody (in our example it is written on a different stave to allow each independent voice to be seen clearly). From this point onwards, both melodies (the original and the shifted copy) develop simultaneously, with a constant overlap. Consider the first four bars:

The two cases of horizontal translation take place in time. A third translation vertically modifies the position on the score: this is called 'transposition'.

A UNIVERSAL LULLABY

The origin of the melody and the words for *Frère Jacques* ("Brother James" in English) is uncertain. There are few doubts that the first written version of the melody under the title "Frère Blaise" was dated at the end of the 13th century. However there are researchers who point out the obvious similarity between the song and the melody of a 1615 work by the Italian Girolamo Frescobaldi. Certain studies have interpreted its words ("Frère Jacques, frère Jacques / Dormez-vous? Dormez-vous? / Sonnez les matines! Sonnez les matines! [...]", which can be translated without repetitions as: "Brother James / Are you sleeping? / Ring the morning bells") as taunting Protestants, Jews or even Martin Luther himself. Others have read it as a reproach to the comfort enjoyed by the Jacobean monks, known in France as living off the clergy. The lullaby, which has been translated into most languages, is so widespread throughout the world that a recent questionnaire among Chinese students found that they believed the song to be part of the popular heritage of their own country!

Vertical translation: transposition

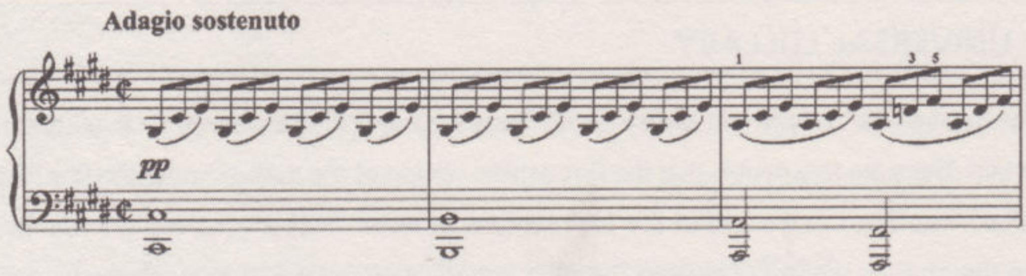
The isometric translation of the notes on the vertical axis generates a transposition. The melody remains the same but in a higher or lower tone, depending on whether it is raised or lowered on the stave.

Original melody

Melody translocated up a 5th

The transposition of a melody is an operation that deserves to be considered in greater depth. The example that has been selected is one that illustrates the most basic principle of how a melody is transposed.

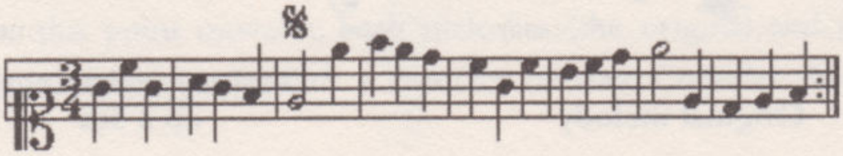
The following examples, taken from a wide range music and styles illustrate some of the uses that are most typical of the symmetry of translation in composition. In *Sonata* op. 27 no. 2, known as *Moonlight*, Ludwig van Beethoven (1770–1827) uses a three-note arpeggio repeated as a texture generator.



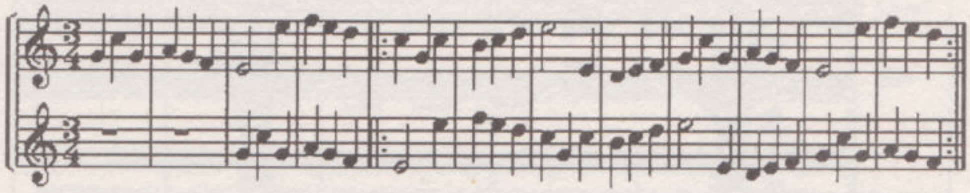
In rock, 'riffs' are short, rhythmical melodies, generally played using a guitar and often repeatedly played one after another. An example of this is *(I Can't Get No) Satisfaction*, by the Rolling Stones, which contains one of the most famous riffs of all time.

The persistent repetition of any musical element (as in the previous two examples) is known as an 'ostinato'.

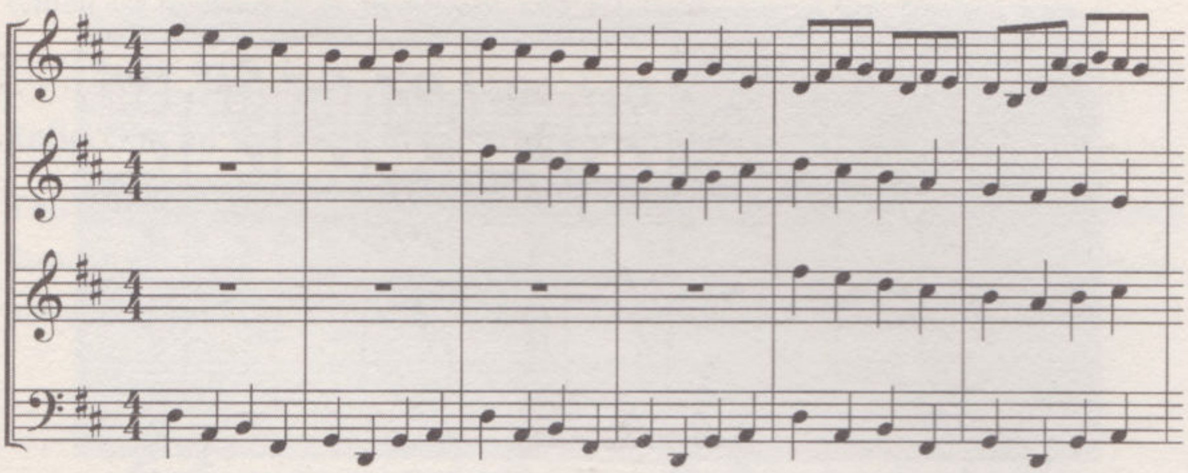
In terms of the more elaborate repetitions of canons, let us consider an example from the grand master in this area, Johann Sebastian Bach (1685–1750). Bach exploited this rather formal system by means of highly ingenious designs. His ability in this art form was such that he was in the habit of giving small, specially written works based on the form as gifts. In the case below, we can see *Canon a 2, Perpetuus* BWV 1075, a short work of eight bars designed to be performed by two voices shifted two bars apart. On account of their structure, the work enters into a cycle that could go on forever:



Here is the development of the canon.

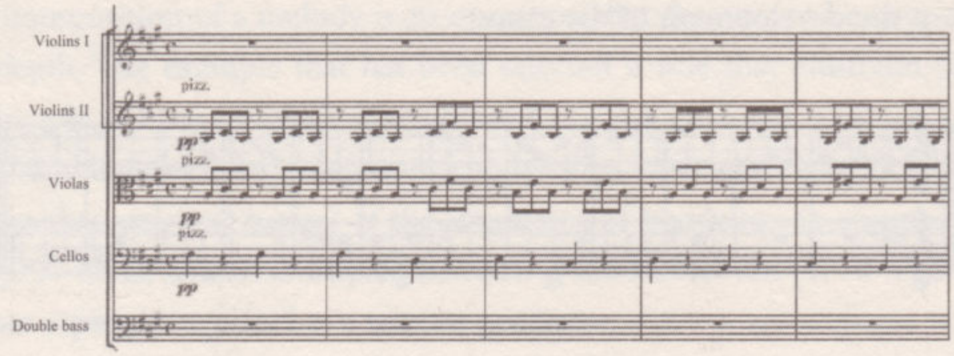


Perhaps one of the most famous canons in history was written by the German composer Johann Pachelbel (1653–1706). His canon in *D* was made famous following its inclusion in the soundtrack of the film *Ordinary People* (1980) and is simultaneously written as a canon and a chaconne, meaning it combines two types of horizontal translation.



The chaconne is a theme with variations in which the bass repeats the same fragment over and over again and the remainder of the voices weave the variations on this theme. In this case, the variations are interpreted by three upper voices in canon in a structure that was commonly used by Pachelbel. Based on a cyclically repeated harmonic cadence (which has been reused many times throughout history), the climax of the piece develops progressively and without jumps, fading in and out of calm and melancholic passages to others that are euphoric or radiant.

Pavana op. 50 by the French composer Gabriel Fauré (1845–1924) is also worth mentioning. The start of the piece includes an initial pattern which is repeated twice (bar 1 and half of bar 2) by the cellos, violas and second violins.

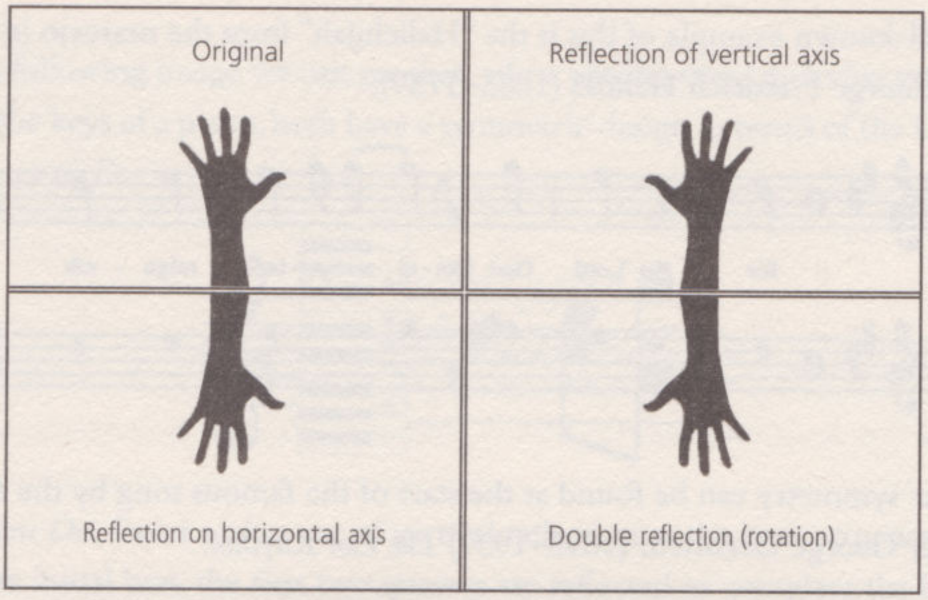


Beethoven's (1770–1827) famous *Symphony No. 5 in C minor* op. 67 provides us with the next example, which contains a diagonal translation that combines horizontal and vertical shifts. In the upper voice (highlighted in the score below), the same one bar melodic expression is repeatedly chained and transposed in an ascending fashion.

Reflections

Reflection is a transformation that modifies the image by inverting it, as if we were seeing its reflection in a mirror. The reflection of a shape generates its mirror image, meaning that we cannot return to the original image by means of a simple rotation. Think of the reflection of the image of a man with a patch on his right eye.

The original mirror can, however, be recovered by carrying out a double reflection, or rather by reflecting the image in a second mirror. We will explore two types of reflections here: the first on a horizontal axis and the second on a vertical one. The combination of both transformations results in a rotation of 180°, as shown in the image:



By applying new reflections to musical patterns, we can obtain new patterns which are referred to as 'inversions' and 'retrogressions' of the original ones.

Reflection on the vertical axis: retrogression

This is when a melody is rewritten starting with the last note, walking backwards through the notes such that those of the original melody are repeated in the opposite direction:

Original melody

Retrograde

Where both the original melody and the retrogression are played one after the other, this is referred to as a case of 'melodic symmetry' which, on account of its horizontal nature, can also be referred to as a 'melodic palindrome'.

Symmetric melody

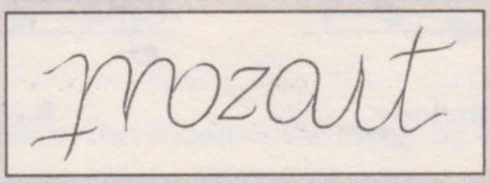
Palindrome

A well-known example of this is the "Hallelujah" from the oratorio in the *Messiah*, by George Friedrich Handel (1685–1759).

The same symmetry can be found at the start of the famous song by the American composer George Gershwin (1898–1937) *I've Got Rhythm*:

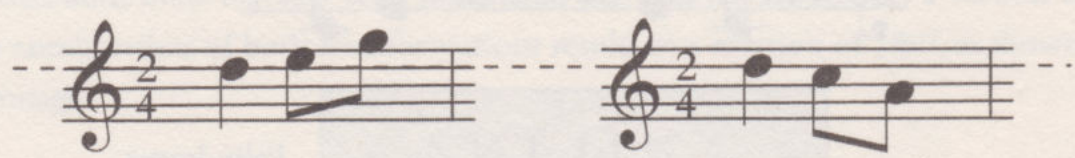
AMBIGRAMS

Palindromic words and numbers are well known instances of symmetries in numbers and letters. Less well known however, are ambigrams: images which represent a text drawn in such a way that applying a transformation (reflection, rotation, etc.) gives rise to the original text or another related one. Here we have one using rotation, based on Mozart's name, by the American Scott Kim.



Reflection on the horizontal axis: inversion

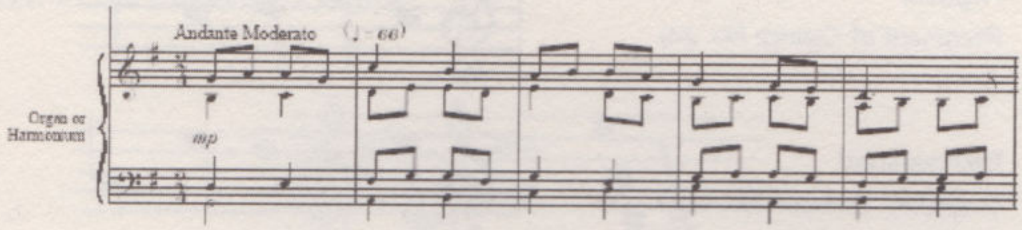
Let us now consider the inversion of a simple melody horizontally reflected on its axis of symmetry on *D*:



Original melody Inversion

In the following image we can see that, when playing these melodies simultaneously on the keys of a piano, both have a symmetric design in terms of the keys used with respect to *D*:

In *Agnus Dei*, Fauré makes use of vertical reflection as the basic structure of this part. In the initial bars, the first two quavers are reflected to complete the bar:



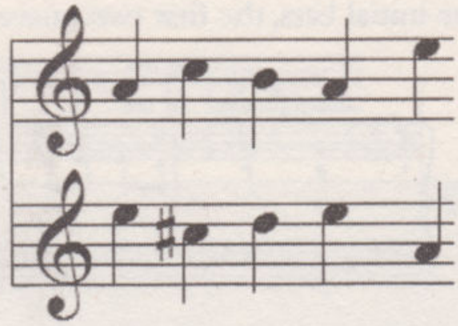
In this passage of the *String Quartet in G Minor* op. 10 by the French composer Claude Debussy (1862–1918), the first violin and the viola continuously alternate opposite (i.e. mirrored) and parallel movement:

The chorus of the *One note samba*, by the Brazilian composer Antonio Carlos Jobim (1927–1994) uses a second bar that is a 180° rotation of the first:

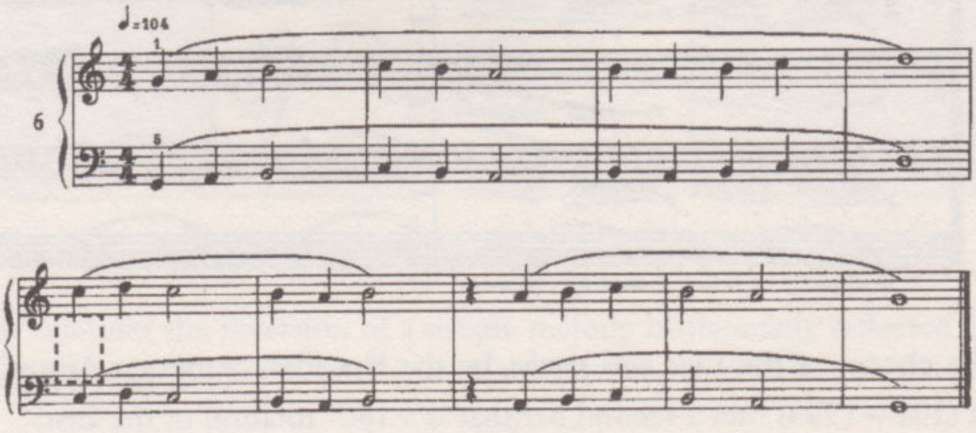
Twenty-four caprices for solo violin by the Italian composer Niccolò Paganini (1782–1840) have inspired a number of variations, perhaps the most famous of which being those by Sergei Rachmaninoff (1873-1943). In particular, *Caprice 24* inspired the Russian composer to create a symmetric version (inversion):

Paganini
(Fragment of *Caprice No. 24*)

Rachmaninoff
(Inversion: *Variation No. 18*)



In some cases, such as the sixth of the "Six unison melodies" of the *Microkosmos* by Béla Bartók (1881-1945), melodies maintain a symmetry in terms of their pitches but not in their durations. The axis of symmetry is on the first note *(C)* of the second system, highlighted on the stave with a dashed box.



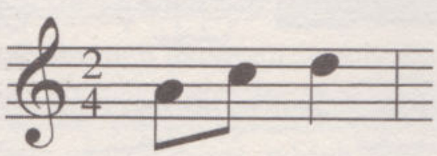
Finally, in the following example, the work of the two hands is symmetric with respect to the initial *B flats*:

Rotations

Recall that a 180° rotation is equivalent to a retrograde inversion. It is the only rotation it makes sense to consider in this text since there is a correlation between the geometrical concept and the perception of the performance of the music exercise. On the other hand, a simple 90° rotation does not make sense musically, as can be seen in the following example:

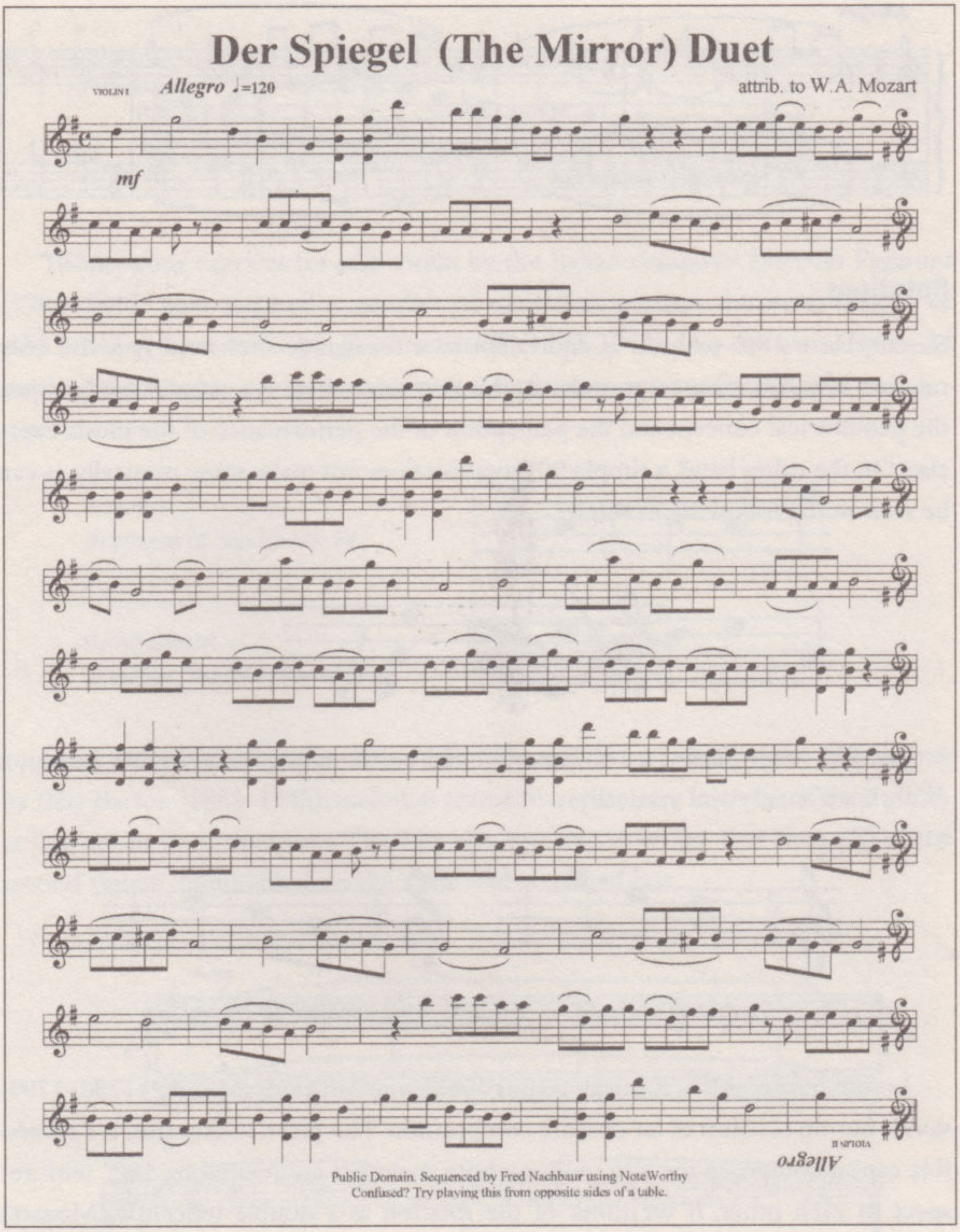
Just like in geometry, a 180° rotation can be thought of as a double inversion (both horizontally and vertically):

Original melody



Retrograde inversion

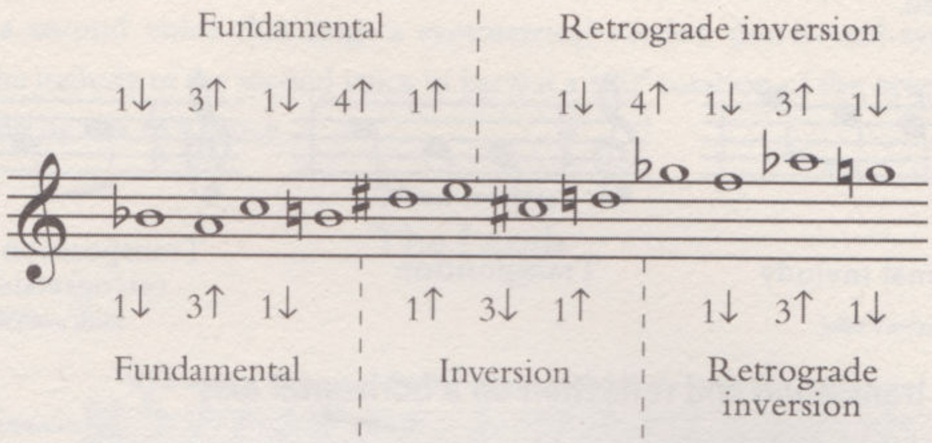
In this respect, the musical genius Wolfgang Amadeus Mozart (1756–1791) stands out on account of an obscure composition. The piece in question is a reversible canon composed for two violins whose melodies are rotated by 180° with respect to each other. If we think of the rotation as a double reflection, Mozart's playful side appears in the horizontal reflection, fixing the axis on the *B* line. This means that it can be written on a single stave with a single melodic line. When it comes to performing the work, the performers face each other with the score placed between them; this is possible thanks to a *treble* clef at each end of the stave: this means that when the page is turned upside down, *G* becomes *D*, *A* becomes *C*, etc, with only the note *B* remaining unchanged:



In Mozart's The Mirror, *two violinists play the inverse of the same score, but in the opposite direction, one sitting across from the other.*

The Austrian composer Anton Webern (1883–1945) is one of the key figures in twelve-tone music, a style that it particularly typical of academic music at the start of the 20th century. In his *String Quartet* op. 28, the composer determines a series of

sounds upon which the intervals are determined as the first step in the composition. In this piece it is possible to observe a fundamental form: retrogression and its inversion. Half-way through, the series presents retrograde and inversion symmetry.



A series of twelve sounds from Anton Webern's String quartet *op. 28. The numbers indicate the semitones of each interval. The arrows indicate if the interval is ascending or descending.*

Combinations

The previously described transformations have been ingeniously combined in various works throughout history. They make up a range of tools available for composition, all the more powerful due to their ability to vary the elements of the transformation – the different locations of the axis of symmetry in reflections, the interval of vertical translations and the horizontal distance between canon parts.

The canon's original idea of playing with the imitation of one voice by another or others has been enriched with other types of imitations, which have also used more original symmetries and retrogressions.

Horizontal translation and vertical translation

The formal essence of the canon consists of the second identical part shifted horizontally. If a vertical shift is also added to this translation, we find a canon the entries of which are spaced at intervals and the second voice begins the melody on a different note from that on which the first starts. This situation requires the modification of certain tones and semi-tones, what is referred to as a 'tonal answer'. The distance at which the second voice enters permits an initial classification of canons.

Vertical translation and reflection on a vertical axis

In this combination, the original melody is first transposed and its retrogression then noted.

Original melody | Transposition | Transposition and retrogression

Vertical translation and reflection on a horizontal axis

To carry out this combination it is necessary to transpose the melody to a new starting note and then write its inversion. However these two operations can be reduced to a single one through the correct choice of the axis of symmetry.

Original melody | Transposition by a 5th | Inverted transposition

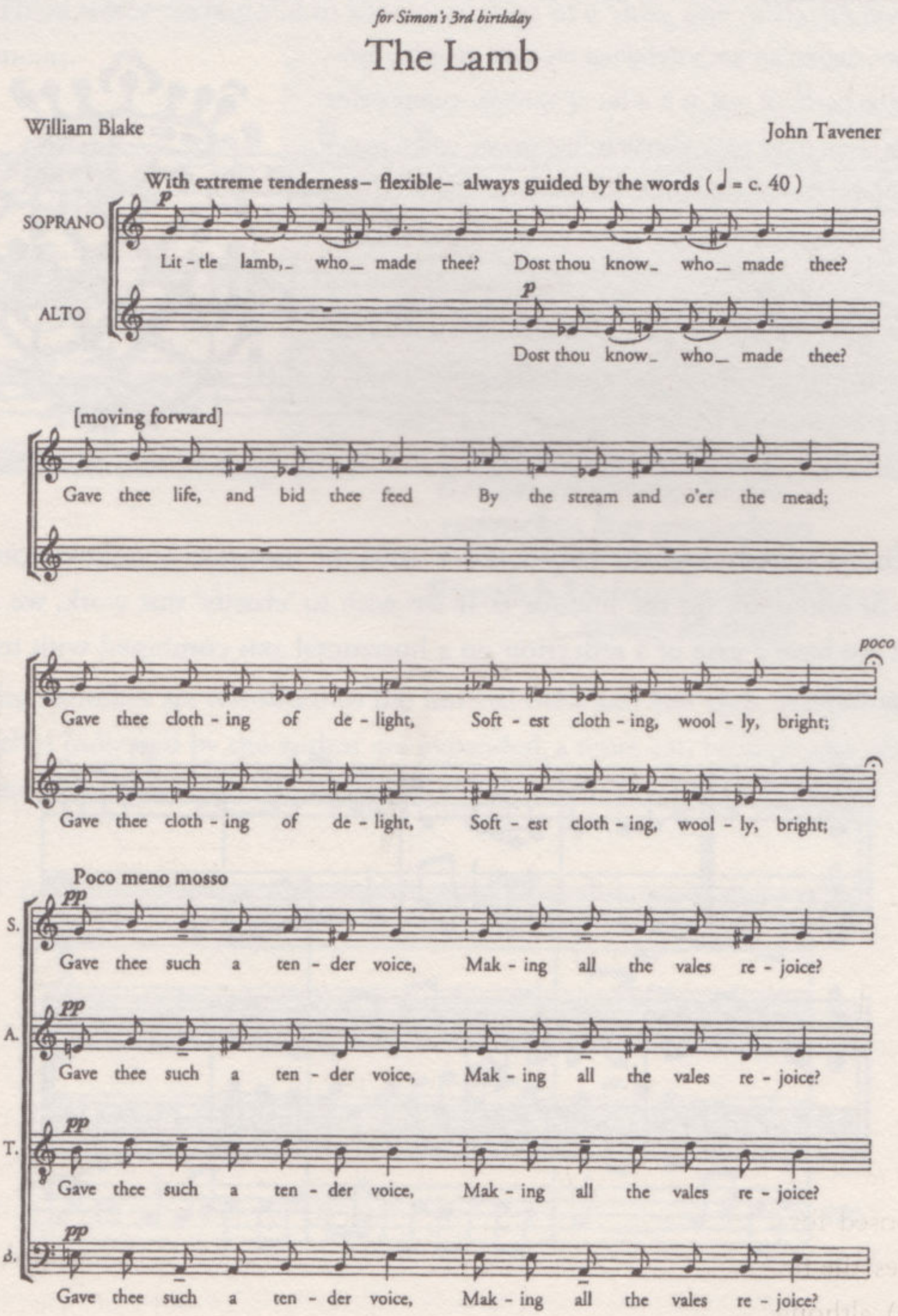
The example shows a vertical translation chained to a horizontal reflection with its axis on B.

Original melody | Inversion

The same result as above but applying a single reflection with respect to an axis on the note G.

The work *The Lamb* by the contemporary English composer John Tavener combines some of the transformations we have discussed in a beautiful choral work composed for his three-year-old nephew. Musically speaking, it is a set of symmetries: the first bar presents a melody which is then repeated in the second (translation), although on this occasion it is accompanied by a second voice that is none

other than an inversion (symmetry on a horizontal axis) of the original one; the axis of symmetry is located on the note *G*. A new melody is presented in the third bar, which is completed in the fourth by a retrograde version (symmetry on a vertical axis); in bars 5 and 6, the melody from bars 3 and 4 is repeated, but with the addition of a second voice that sings a symmetrical version (horizontal symmetry). Hence the melody of the second voice in bar 6 is a 180° rotation of the original from bar 4 sung by the first voice.

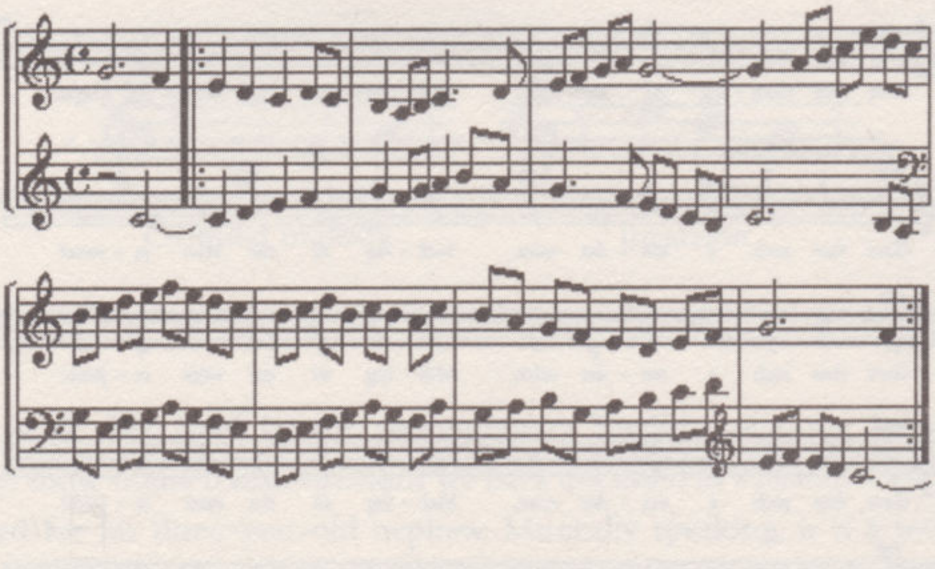


Even if the intervals between each pair of notes in the melody are strictly respected, the composer modifies the duration of the last note of each phrase for aesthetic reasons. However the effect of symmetry is not lost, given that the melodic line is constructed in the mind like a drawing which results from joining the attacks of the sounds, independent of the sustain and fall of the notes.

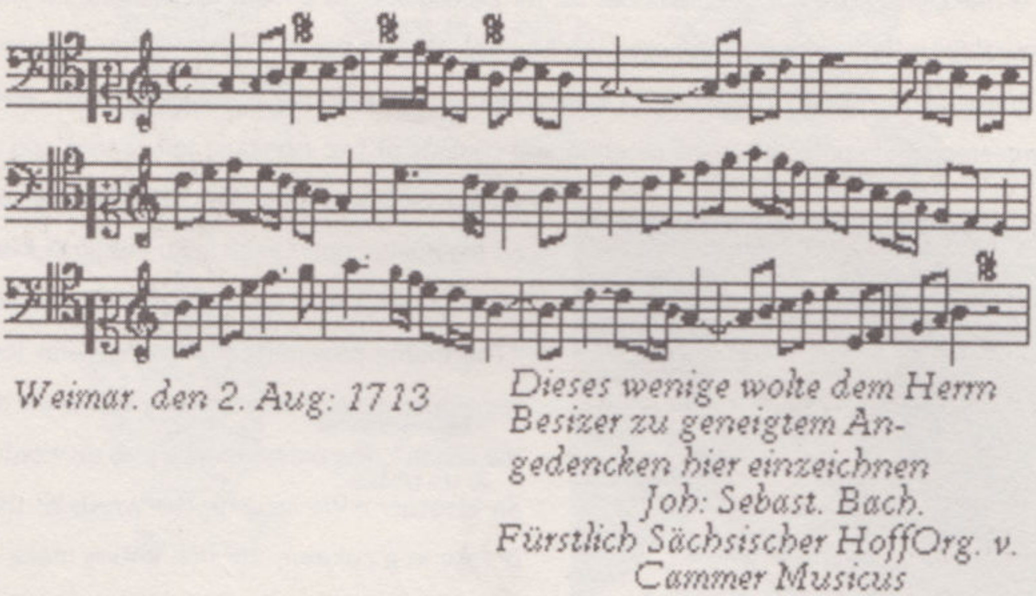
THE SEAL OF BACH

Johann Sebastian Bach designed his own perfectly symmetrical personal seal. It is a set of symbolic components made up of three basic elements: the crown, which represents God; the three initials of Bach's name, JSB, which are arranged vertically with their mirror images; finally, combining the J of one symmetry with the S of the other gives the Greek letter χ, which represents the cross, and is the initial of Christ in Greek. This arises in the two symmetries and a third time when the two Ss combine.

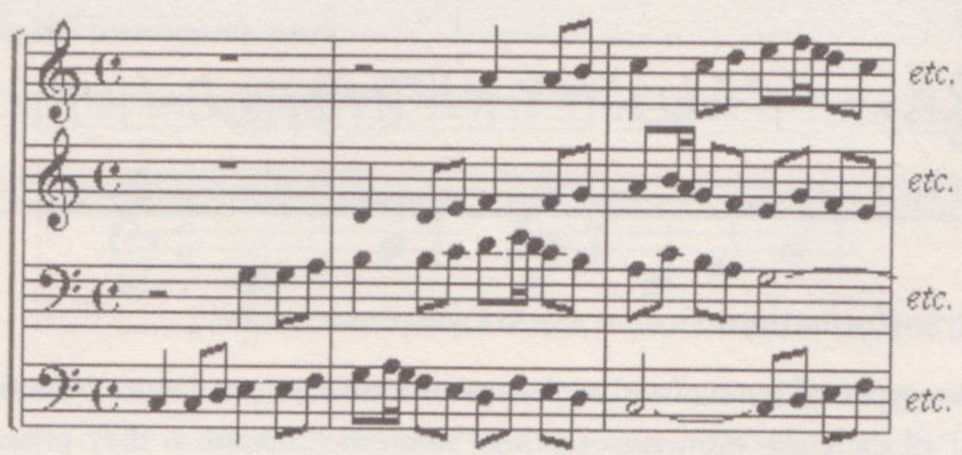
In Bach's *Canon Concordia Discors*, BWV 1086 the imitation is an inversion with an axis of symmetry on the line for *E*. If we wish to 'classify' this work, we could say that we have a case of a reflection on a horizontal axis combined with translation (canon).



Another feature of some of Bach's creations is the cryptic way in which they are set out, making it necessary to first decipher the instructions in order to be able to perform the canon. In the piece with catalogue number BWV 1073, the composer writes a single melody on the stave and provides not one but four clefs. Each provides a different value for the notes on the stave in such a way that the melody is interpreted in a different way for each clef. Following the order in which the clefs are presented, the melody begins on a *C*, then on a *G*, then on a *D* and finally on an *A*. These notes correspond to the four strings of a viola, one of Bach's favourite instruments.



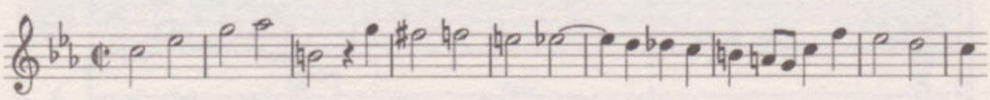
If the melodies are translated to the normal *treble* and *bass* clefs, and the successive entries indicated by the author are expanded, a score can be arranged with the complete development of the canon.



THE KING'S THEME

In 1740, Carl Philipp Emanuel Bach (1714–1788), Johann Sebastian's fifth child, joined the court of Frederick the Great, king of Prussia, in whose palace daily chamber music concertos were given. The king, a music lover, flautist and also a composer, had heard about the art of Bach senior and wished to meet him. After much persistence, his son Carl managed to get Bach to accept the invitation. During his stay in Postdam, the location of the palace, at the request of the king, Bach tried all the Silbermann pianos he had in the chambers and salons. Showing his creative capacity, during his tour, Bach asked the king, completely unexpectedly, for a melody to build a fugue. Bach set off for Leipzig and, as a mark of gratitude for the hospitality he had received, composed the *Musical Offering* based on the melody provided by Frederick the Great. The work, in which Bach so marvellously developed his art, was completed two months after the meeting and consists of two ricercars, ten canons and a sonata. Bach wrote the title of the first ricercar on the manuscript: "Regis Iussu Cantio Et Reliqua Canonica Arte Resoluta", which means "The theme provided by the king, with the remainder developed according to the art of the canon". The phrase hides a play on words: an acrostic. If we arrange the words of the phrase in a column, the first letters make a word when read vertically: RICERCAR.

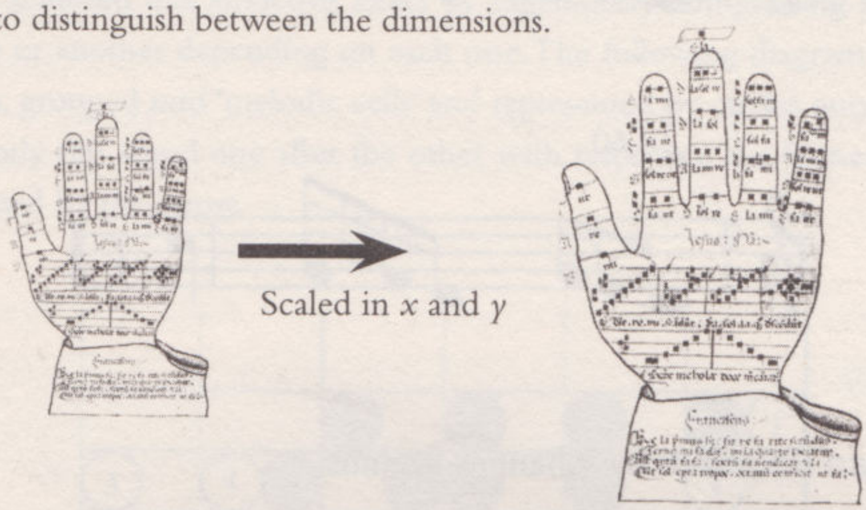
A portrait of Carl Philipp Emanuel Bach and below, the king's theme for Frederick the Great.

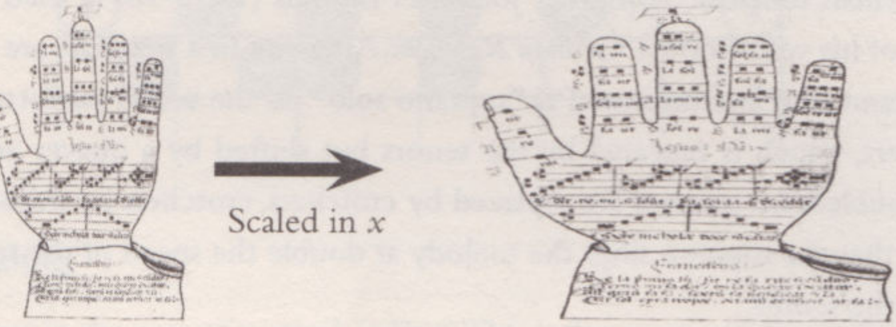


Scalar transformations

The three types of symmetries we have seen thus far (translation, reflection and rotation) all share the property of being "isometric" insofar as they preserve the original sizes and differences between each of the musical elements.

There is also a non-isometric transformation that has its musical correlative: 'scaling'. This increases or decreases the measure of a note in one of its dimensions. Depending on the way in which the scaling is applied, the proportion of the note will either be retained or deformed. Applying this to the realm of music, it is first necessary to distinguish between the dimensions.



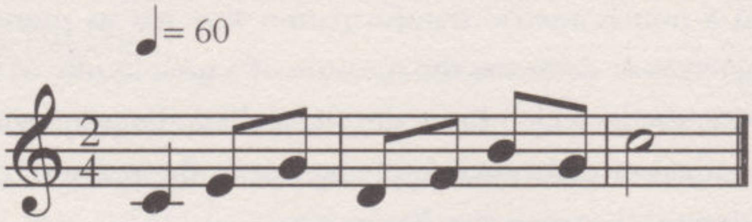


Horizontal scaling

The clearest examples are those which only apply the scaling to the axis representing time. One way of modifying this scaling and thus the speed of a work, is by changing the metronome mark:

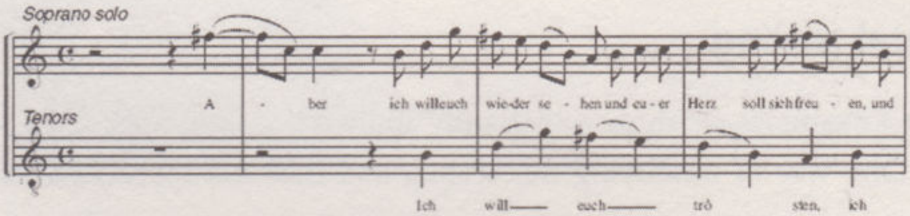
Change of tempo using a change of metronome.

However, it is often more attractive to proportionally modify the speed of the performance while maintaining the same beat (the same metronome mark). To do so, the notes are replaced by their equivalents of greater or lesser duration:



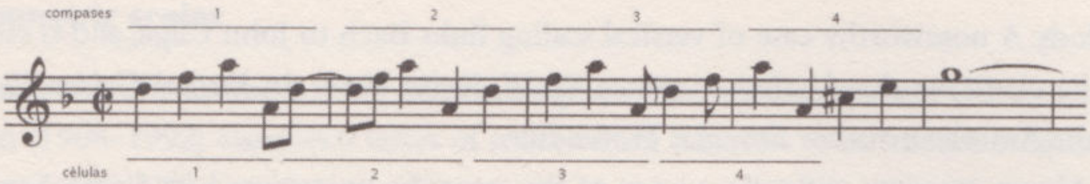
The *German Requiem* by Johannes Brahms

The German romantic composer Johannes Brahms (1833–1897) used scaling in a passage of his well-known *German Requiem*. After the first few bars are played solo, the soprano (the line indicated as "soprano solo" on the score) performs a melody in quavers, which is repeated by the tenors but shifted by a quaver and with the notes doubled: i.e. quavers are replaced by crotchets, crotchets by minims, etc. The result is that the soprano sings the melody at double the speed of the tenors ("tenors" on the score):

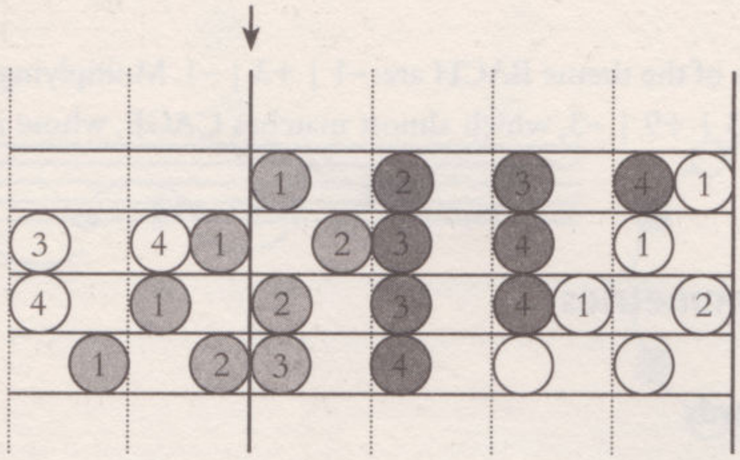


"Puttin' on the Ritz"

This well known melody is the work of the American musician Irving Berlin (1888–1989), "the greatest songwriter that has ever lived", in the words of his fellow countryman George Gershwin. It was originally published in 1929 and was later recorded by Benny Goodman and Fred Astaire, among others. In spite of being a popular composition, the rhythm of its verses is of considerable complexity. The melody repeats a basic four-note cell four times, although the four repetitions do not occupy four bars but just over three, achieving a disconcerting rhythmical effect:

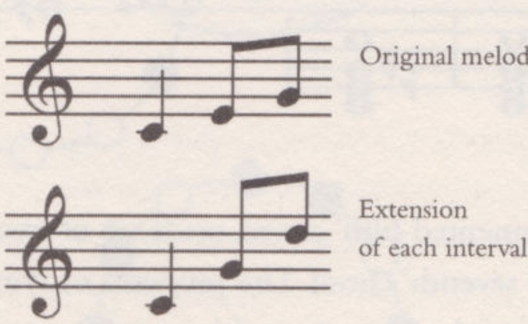


Berlin achieved this attractive effect by ingeniously compressing some of the notes, one or another depending on each case. The following diagram shows how four notes, grouped into 'melodic cells' and represented by circles numbered from 1 to 4, subtly slip ahead one after the other with respect to the framework of the bar, indicated by the arrow.



Vertical scaling

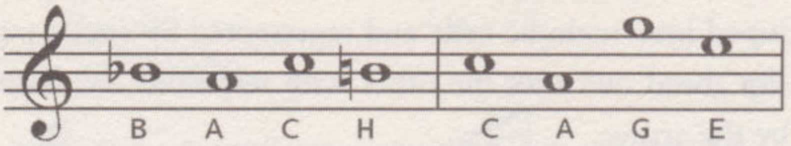
What happens in the case of vertical scaling? This is the most curious case of all, the least feasible and musically speaking, the least recognisable. In a vertical scaling transformation, all the intervals are proportionally amplified. In the first example, the intervals of the first melody are two thirds; in the second, these thirds are transformed into fifths.



The aim of repeating the original but expanded melodic curve can be achieved in certain cases, but this transformation can also result in a parody of the original

melody. A noteworthy case of vertical scaling links Bach to John Cage, and is cited in the classic work of popular science *Gödel, Escher, Bach: An Eternal Golden Braid*, by the American author Douglas Hofstadter.

Always starting with the names of the notes as represented in Anglo-Saxon notation, from A to G, it is possible to scale the theme BACH to give CAGE… or almost.

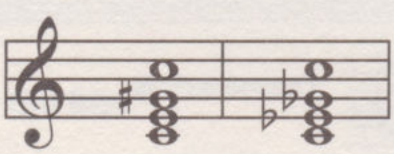


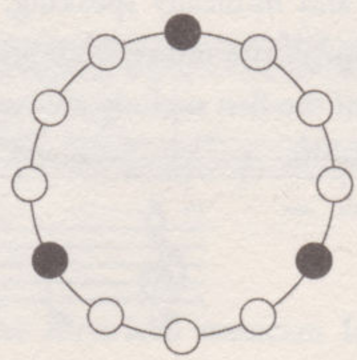
The intervals of the theme BACH are: –1 | +3 | –1. Multiplying these intervals by 3, we have –3 | +9 | –3, which almost matches CAGE, whose intervals are –3 | +10 | –3.

Harmonic symmetries

Symmetric chords

The space of an octave consists of twelve semitones. There are only two ways of symmetrically distributing this 12 semitone space: one with three notes with four semitones between them and another with four notes and three semitones between them.



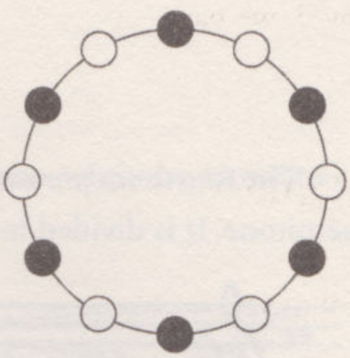


The first is the 'augmented fifth' chord, made up of two major thirds. The second is the 'diminished seventh' chord. The property of symmetry of this chord in particular has meant that it has been one of the great figures in the history of music, since any appearance of the chord can be 'read' in many ways at once.

Symmetric scales

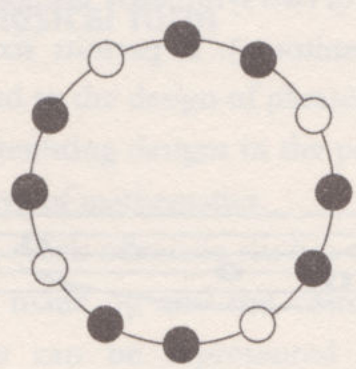
In his book *The Technique of My Musical Language*, the French composer Olivier Messiaen (1908–1992) classifies a series of scales which he refers to as "modes of limited transposition". These scales span the octave and symmetrically distribute the intervals that separate each pair of notes. They are based on the chromatic system of twelve sounds and are made up of different symmetric groups. Once the scale has been established, it is successively transposed until it is impossible to continue. This occurs when a new transposition generates a scale that repeats the notes of the first group.

The first scale of Messiaen's classification is referred to as the 'whole tone scale'.

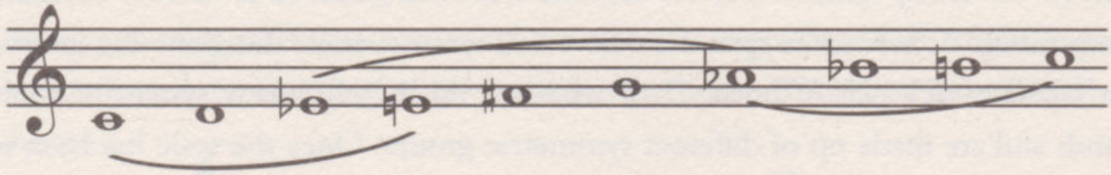


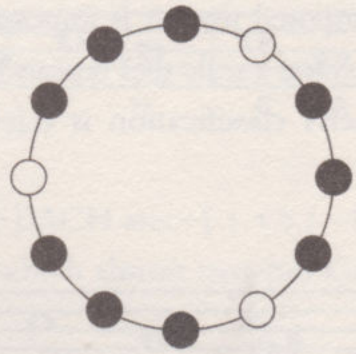
This scale only allows two arrangements, starting from *C* and *C* sharp. The one starting on *D*, repeats the notes of the initial scale.

The second scale is the diminished octatonic scale which alternatively links semitones and whole tones. It is divided into four groups of three notes and allows three transpositions.

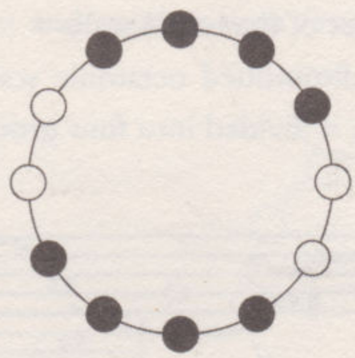


The third scale contains tone-semitone-semitone and makes three groups of four sounds. It permits four transpositions.

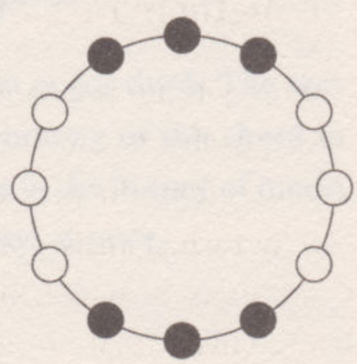




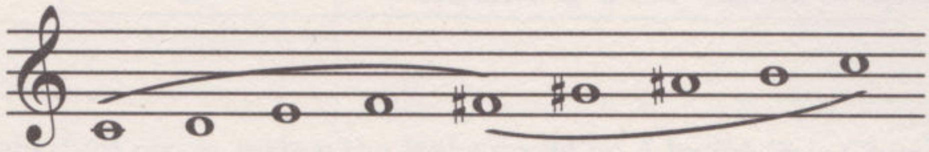
The fourth scale contains semitone-semitone-tone and a half (three semitones)-semitone. It is divided into two groups and permits six transpositions.

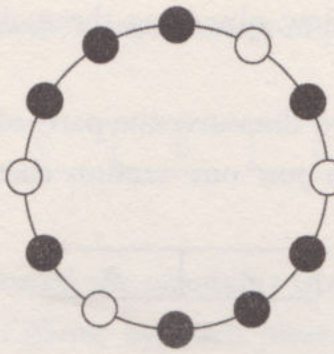


The fifth scale is made up of two symmetric groups (semitone-two tones-four semitones). It permits six transpositions.

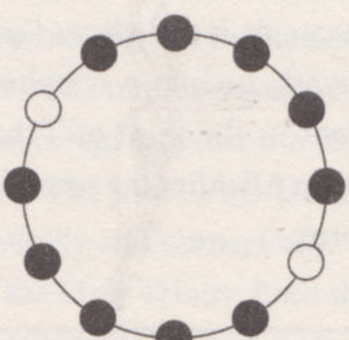


The sixth scale is divided into two groups of five sounds (tone-tone-semitone-semitone) and has six transpositions.





The seventh scale has two groups of six sounds (semitone-semitone-semitone-tone-semitone) and has six transpositions.



The mathematics of musical form

Symmetry is not just confined to the design of phrases and melodic patterns, but it is also possible to observe interesting designs in the parts of more complex formal structures from the perspective of mathematics.

The formal analysis of a work of music studies the 'plane of the work', that is, the parts of which it is made up and the connections that exist between them. Given that a plane can be represented with different degrees of

approximation, depending on the scaling that is used, it is either possible to achieve a general view, with the concomitant loss of details, or the focus can be on the details at the cost of the general view.

ABCDE...

Let us begin with a general view, observing the main structures, which we shall designate using capital letters.

In the context of an ordered discourse the parts of form are defined by repetition or change. A work with just one section that is repeated whole can be expressed as:

A	A

which is a simple case of symmetry. A work with two completely different sections, on the other hand, does not give any symmetry:

A	B

Do formally symmetric works exist? The answer is 'yes'. Moreover, this is a commonly occurring form. One example is the *scherzo* ('game' or 'joke', a musical form that is often part of a larger work, such as a symphony (for example, the second movement of Beethoven's *Ninth*, or the third of Tchaikovsky's *Fourth*). Essentially, the *scherzo* is binary, with the form AB. After the second part has been played, the first is repeated, giving a ternary form:

A	B	A

Naturally, this is the simplest form of symmetry. The repetition can be repeated, thus generating more symmetric forms.

A	B	A	B	A

There are also composite ternary forms, with each part made up of ternary forms. This results in larger symmetric structures:

A	B	A
aba	cdc	aba

Certain short pieces, such as the *Waltz* op. 34 no. 1 by Frédéric Chopin (1810–1849), include even more extensive symmetries in part of their structure:

A	B	C	D	C	B	A

As longer works are structured, the possibility that the second part is a retrogression decreases. The *Musical Offering* by Bach presents a formal symmetry of the following sort:

RICERCAR	CANONS	SONATA TRIO	CANONS	RICERCAR

Bach's *Mass in B minor*

In a number of his works, Bach, the most formally inventive composer in history, makes use of structures with symbolic and mathematical properties. His *Mass in B minor* BWV 232, is composed of 27 numbers grouped in four parts: "Kyrie", "Gloria", "Credo" and a fourth that brings together the sub-parts "Sanctus", "Hosanna", "Benedictus" and "Agnus Dei". The idea of the German composer was to symbolise the Holy Trinity both musically and numerically.

The number 3 represents the Holy Trinity. Both the total of the numbers of the work (27) and each of the distributions in the four parts (3 + 9 + 9 + 6) is divisible by 3. In particular, the central numbers ("Gloria" and "Credo") have a symmetrical structure. "Gloria" has its centre of symmetry in the section "Domine Deus" ("Lord God"). In "Credo" it is on "Crucifixus" ("The Crucufuction"):

—Kyrie

Kyrie eleison (1st).

Christe eleison.

Kyrie eleison (2nd).

—Gloria

Gloria in excelsis.

Et in terra pax.

Laudamus te.

Gratias agimus tibi.

Domine Deus. ←

Qui tollis peccata mundi.

Qui sedes ad dexteram Patris.

Quoniam tu solus sanctus.

Cum Sancto Spiritu.

—Credo

Credo in unum Deum.

Patrem omnipotentem.

Et in unum Dominum.

Et incarnatus est.

Crucifixus. ←

Et resurrexit.

Et in Spiritum Sanctum.

Confiteor.

Et expecto.

—Sanctus, Hosanna, Benedictus, and Agnus Dei

Sanctus.

Hosanna.

Benedictus. Aria

Hosanna (da capo).

Agnus Dei.

Dona nobis pacem.

In the "Credo" in particular, the three central elements summarise the life of Christ, from his conception *(Et incarnatus est)* all the way to the resurrection *(Et resurrexit)*, with the crucifixion *(Crucifixus)* as the central movement.

MUSICAL CRYPTOGRAMS

A cryptogram is a message that cannot be understood unless the recipient knows the key. The message can be hidden in a drawing, a text or a jumble of numbers and letters. In all cases the meaning is restored by using a known code. A musical cryptogram is a piece that makes it possible to decipher a text by simply naming the notes that are heard. Many composers have used this system to create melodic patterns. The most famous is without doubt B-A-C-H, which uses the classical German naming convention, in which *B flat* is represented by the letter B, *A* by A, *C* by C, and *B* by H.

Other famous patterns are:

- ABEGG: in homage to Meta Abegg, in the *Abegg Variations,* by Robert Schumann.
- CAGE: for John Cage, used by Paulin Oliveros.
- GADE: for Niels Gade, used by Robert Schumann.

Anton Webern, in the series that appears in his *String quartet* op. 28, selects the four notes BACH, and uses two geometrical transformations to construct the other eight notes based on this group.

For his part, the Austrian Alban Berg (1885–1935), in his opera *Wozzeck*, pays homage to the three main representatives of the Viennese school, creating a cryptographic design for each instrument:

- The piano: Arnold Schönberg (ADSCHBEG).
- The violin: Anton Webern (AEBE).
- The trumpet: Alban Berg (ABABEG).

The golden ratio and music

The Italian Leonardo de Pisa, better known as Fibonacci (1170?–1250) was one of the figures who introduced Arabic numeral notation to the West. In his work *Liber Abaci* he formulates the following problem:

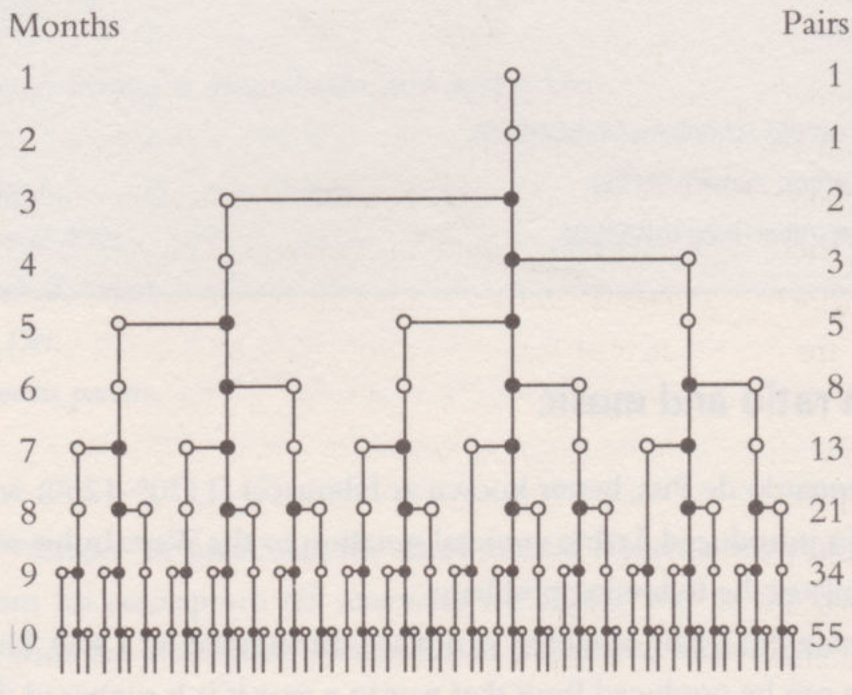
A certain man put a pair of rabbits in a place surrounded by a wall. How many pairs of rabbits can be produced from that pair in a year if it is supposed that every month each pair begins a new pair, which, from the second month on, also becomes sexually reproductive?

The solution to this intriguing question is as follows:

- For the first two months, there is only one pair of rabbits (A).
- During the third month, the first pair of A's descendants is born (B).
- In the fourth month, the second pair of A's descendants is born (C).
- In the fifth month, the third pair of A's descendants is born (D), as well as E, the first pair of B's descendants.

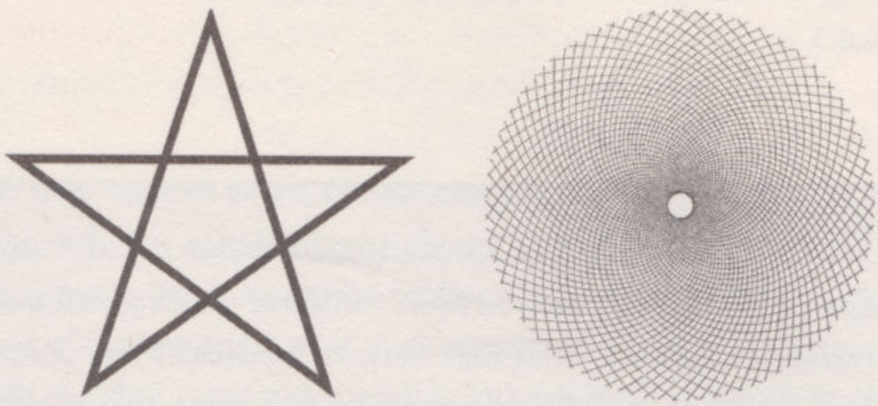
Taken as a sequence over the months, the number of pairs of rabbits is: 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144... This sequence of numbers is known as the 'Fibonacci series'. If we calculate the quotients of the neighbouring terms of this sequence, we obtain the following values:

$$1/1 = 1$$
$$2/1 = 2$$
$$3/2 = 1.5$$
$$5/3 = 1.666...$$
$$8/5 = 1.6$$
$$13/8 = 1.625$$
$$21/13 = 1.615...$$
$$34/21 = 1.619...$$



A diagram showing the growth of a population of rabbits. The white dots indicate pairs of young rabbits and the black dots indicate they are adults capable of reproducing.

The limit of this series of quotients is none other than 1.618033989…, the number that is referred to as the 'golden ratio', also known as the 'divine proportion' or the 'golden section', and which has long been associated with harmony and beauty. The terms of the Fibonacci sequence emerge in many diverse areas of nature, such as the arrangement of seeds in sunflowers, the angle made by the leaves of plants at different heights on the stalk, the spiral of the shells of snails…



All the segments of the five-point star, or 'pentagram', which has a strong symbolic presence in many cultures, respect the golden ratio. To the right, a diagram of the arrangement of the seeds of a sunflower, with 55 spirals in a clockwise direction and 89 in an anticlockwise direction.

Music has not escaped the attraction of the golden ratio. Certain compositions by Mozart and Beethoven appear to reach their climax, the moment of maximum tension, in a point that divides the work into sections with lengths that are approximately determined by the golden ratio. It is most likely that both geniuses reached this result intuitively, seeking formal equilibrium in their music. In the case of Bartók however, Fibonacci numbers appear so clearly in his work that it is unlikely that he was not using them deliberately. Hence the initial fugue of his *Music for Strings, Percussion and Celesta* has its (almost) 89 bars divided into sections of 55 and 34. In turn, these sections are also subdivided according to the Fibonacci numbers: the first into 34 and 21 bars and the second into 13 and 21. The third movement of the work, an adagio, begins with a rhythmic progression in which the xylophone plays the same *F* in groups of 1, 1, 2, 3, 5, 8, 5, 3, 2, 1 and 1. His *String Quartet No. 4* has a total of 2,584 beats in its five movements, the 18th number in the Fibonacci sequence.

The Fibonacci numbers can also be found behind certain interval models used by Bartók, which include intervals made up of 2, 3, 5, 8 and 13 semitones.

Similarly, certain works by Debussy would appear to be organised according to the golden ratio or at least in line with the Fibonacci numbers. The introduction of "Dialogue du vent et la mer", of the orchestral work *La Mer*, has 55 bars subdi-

vided into sections of 21, 8, 8, 5 and 13 bars. The 'golden' bar, 34, is marked by a burst of the trombones.

Although many of these analyses offer a degree of approximation to reality, it is best to take them with a pinch of salt. There is an element of a self-fulfilling prophecy. It common for a listener who has been previously 'warned' of the (possible) presence of golden ratios to feel somewhat unimpressed by what they hear.

THE MEASURE OF BEAUTY

In the creative process, the artist creates the form – the detailed and the general, seeking tensions and relaxations, edges and curves, highs and lows. The result is a state of equilibrium, be it stable or unstable. The aesthetic pleasure that can be evoked in a viewer by a work of art is ultimately a subjective matter. However, is there a way to at least approximate objective criteria of beauty? The golden ratio is perhaps the best known of the objective means for gauging the beauty of an object, although there have been other attempts. This was the case with the American mathematician George Birkhoff (1884–1944). After fieldwork in which he studied various artistic manifestations, at the start of the 1930s he published *A Mathematical Theory of Aesthetics* and *Aesthetic Measure*. Birkhoff's studies were focused on music and poetry. He defined a factor referred to as the aesthetic measure that involves two components: the 'aesthetic order' (O) and the 'complexity' (C):

$$M=\frac{O}{C}.$$

The aesthetic order is given by the regularity of the elements which make up the object being studied, while the complexity measures the degree to which these elements are present. Birkhoff was the first to recognise that to obtain representative results he could not study the work as a whole, but a certain property of the object being studied. In the case of music, for example, the isolated chords of the rhythm and the harmonic context. Birkhoff dedicated three chapters of the book to music, in which he analysed chords, harmony, melody and counterpoint. Regardless of the effectiveness of the system, it is interesting to note that the equation suggests that a work is more beautiful when it is less complex, that is to say there is a direct relationship between beauty and simplicity.

Chapter 4

Waves and Bits

Music is the arithmetic of sounds as optics is the geometry of light.
Claude Debussy

Let us now tune our ears to the various properties of sound and as the relationships between the different strands emerge plunge further into their hidden depths. This approach to sound, not as an artistic act but as a physical phenomenon, requires us to make use of mathematical tools in an attempt to decipher it.

Similarly, we shall submerge ourselves in torrents of electrons in electrical circuits in order to understand how audio information moves when it does not take the form of sound.

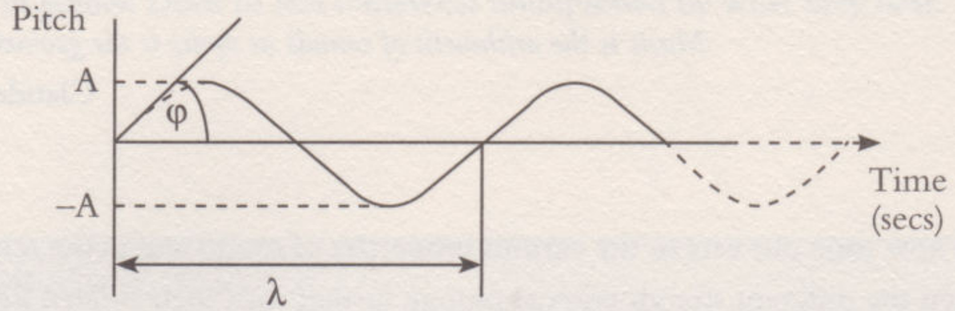
The physics of sound

We have been focusing on a property of sound that we refer to as pitch or tone, associated with frequencies of vibration. A sound is generated by the oscillating motion of a solid body, be it of metal, wood, skin or even the vocal chords or a flow of air or water. Once the vibration has been emitted, it is propagated to the neighbouring particles.

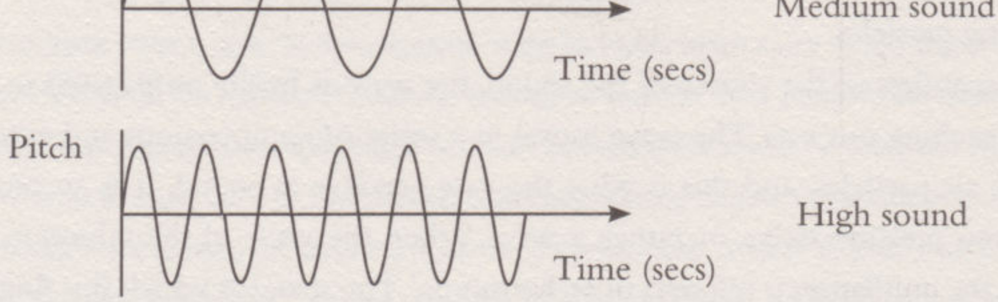
Regardless of the source of the sound, the wave is finally propagated to the air, thus reaching our ears. The wave moves as a series of compressions and expansions of the air particles, and this is what the ears perceive as sound. It is an oscillation between pressure states, or rather, a wave. When the wave of the vibration is uniform, the oscillation is referred to as 'harmonic'. The speed at which this fluctuation between the two states takes place is referred to as the 'frequency' and is measured by the number of oscillations per second (Hertz). The greater the frequency, the higher the pitch of the sound that is perceived.

An audible oscillation starts from a state of rest and gradually grows until it reaches its maximum height (A), at which point it begins to return to the state of

rest in order to begin the oscillation in the opposite direction until reaching a new maximum height (–A). Upon returning to the rest point, it will have completed a full cycle (λ). At this point, the gradient of the curve is the same as at the starting point. Mathematically, the oscillation of a pure sound is represented using the sine function:



Each of the variables of this function is associated with one of the properties of the sound: pitch, intensity and timbre. The 'pitch' is determined by the frequency of the oscillation, such that a low frequency of oscillation corresponds to low tones and a high frequency of oscillation, to high tones.

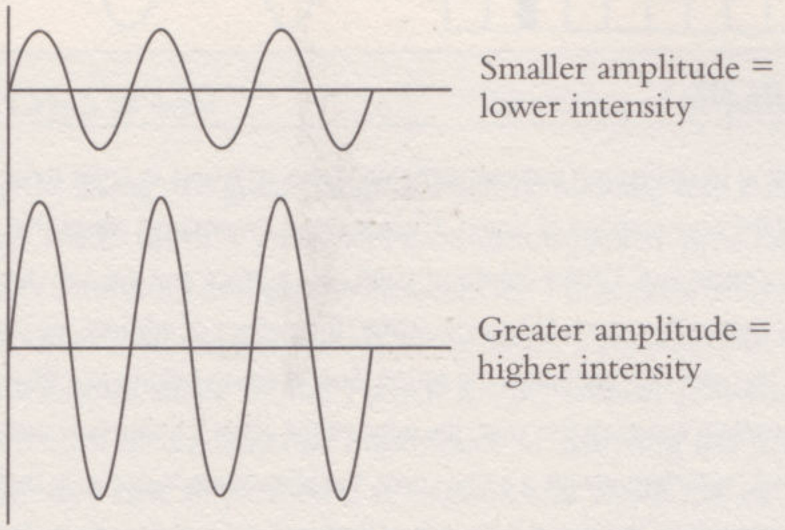


The pitch of the sound is proportional to its frequency.

The spectrum of frequencies which can be detected by the human ear varies from person to person and depends on age. Generally speaking, however, it spans eleven octaves:

1st octave:	16–32 Hz
2nd octave:	32–64 Hz
3rd octave:	64–125 Hz
4th octave:	125–250 Hz
5th octave:	250–500 Hz
6th octave:	500–1,000 Hz
7th octave:	1,000–2,000 Hz
8th octave:	2,000–4,000 Hz
9th octave:	4,000–8,000 Hz
10th octave:	8,000–16,000 Hz
11th octave:	16,000–32,000 Hz

The 'intensity', or rather the acoustic energy developed by a wavelength per unit of time depends on its amplitude, hence the greater the amplitude of the wave, the higher the volume. It is interesting to note that the hearing threshold is stated in terms of acoustic intensity (2×10^{-4} bar). The maximum tolerance (i.e. the pain threshold) is at around 200 bar.



The intensity of the sound is proportional to its amplitude.

The unit of measurement for acoustic power is the bel, although the decibel (dB), one tenth of a bel, is used in practice. Its design takes account of the fact that the relationship between the sensitivity of the human ear in terms of intensity is approximately logarithmic. The scale of intensity starts from 0 dB, the threshold of hearing, and goes up to 120 or 140 dB, the pain threshold.

The following table provides some examples of activities that generate noise and an approximation of their intensity:

Intensity of the sound	
120-140 dB	Pain threshold
120 dB	Aeroplane motor running
100 dB	An orchestra playing
90 dB	Noisy street, lots of traffic
80 dB	Train
70 dB	Metal orchestra
50 dB	String orchestra
40 dB	Conversation
20 dB	Reading room
10 dB	Relaxed breathing
0 dB	Hearing threshold

WAVES IN 3D

It is interesting to distinguish between different types of waves in order to be able to better understand the phenomenon of sound. There are one-dimensional waves that transport their pulses in a straight line. Others are transmitted on a surface and are two dimensional, such as the ones caused by a stone falling into water; their edges of advance are concentric circles centred on the spot that generates the sound. Sound waves belong to a third group, that of three-dimensional waves; in this case, the edge of the wave is a spherical surface. Even if the equation that represents waves is a sine curve, the phenomenon occurs in three-dimensional space. The intensity of the sound is the power exercised per unit of surface. As we are dealing with a series of concentric surfaces, intensity is measured using the following formula:

$$I=\frac{P}{S},$$

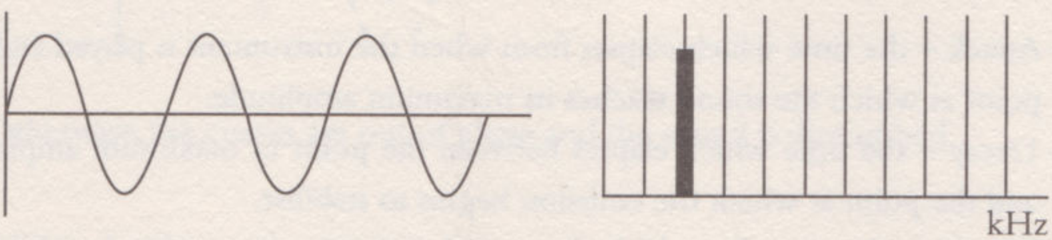
where I is the intensity, P the power and S the surface. Given that $S=4\times\pi\times r^2$, the intensity is inversely proportional to the square of the distance.

Finally, the 'timbre' of a sound gives it its 'personality': the quality that, for example, makes it possible to identify the voice of a given person. Timbre also makes it possible to distinguish between sounds emitted by different instruments, although they may have the same intensity and pitch. But what is the physical basis of timbre? To answer this question, it is necessary to study the nature of sounds in greater depth.

Pure and real tone

The graph of the sine wave represents the oscillation of a pure sound, something which is uncommon in the real world. Examples of pure sounds include those generated by a tuning fork, a whistle, or the sound produced by rubbing the edge of a wine glass with a moistened finger.

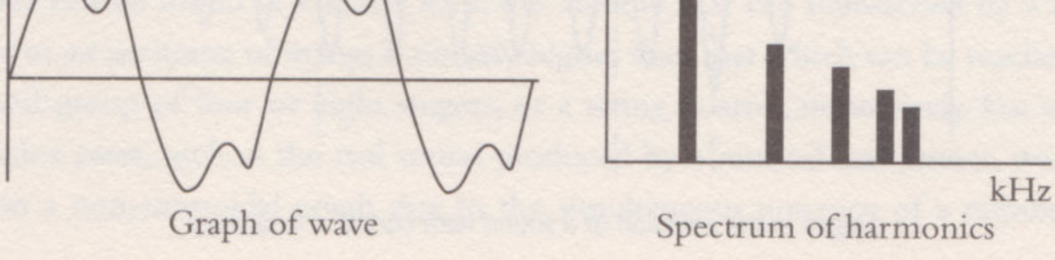
Pure sound



Graph of wave — Spectrum of frequencies

But plucking the string of a guitar, ringing a bell, or blowing into a flute generates a complex sound made up of a main vibration accompanied by a range of other waves with smaller intensities and greater frequencies. These associated waves are referred to as 'harmonics". All impure sounds are, in short, a collection of simultaneous sounds. The contribution made by the French mathematician Jean-Baptiste Joseph Fourier (1768–1830) is of fundamental importance in analysing the harmonies of a sound. Fourier showed that all periodical non-sinusoidal waves can be decomposed into a series of sine waves.

Impure sound

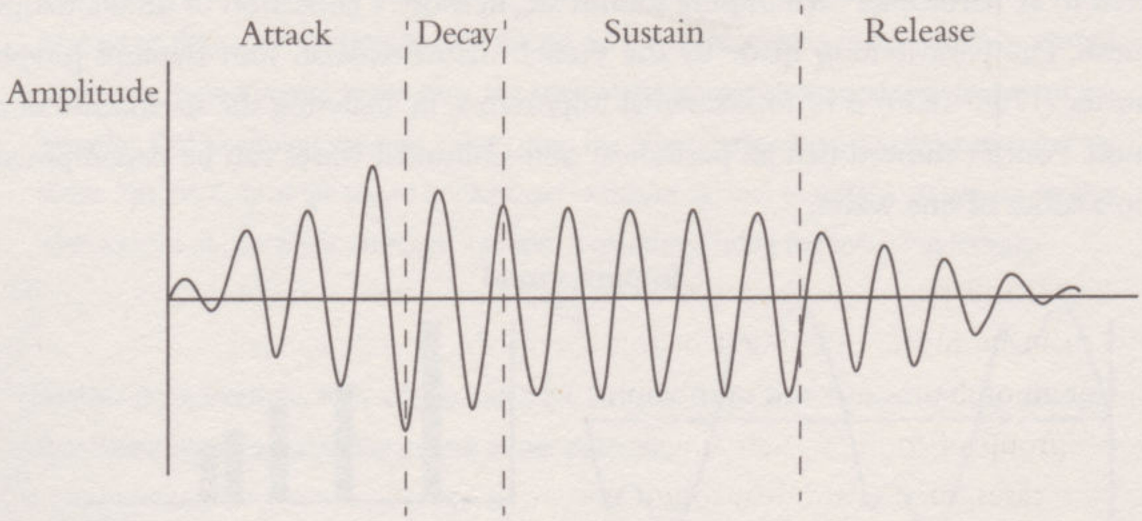


Graph of wave — Spectrum of harmonics

In the case of sound, it is possible to distinguish the audio range of a wave for each harmonic and another for the fundamental sound. This apparent chaos in fact follows a highly ordered system. The very nature of the material generated, the surrounding medium, its resonance, etc., affects how the fundamental tone generates the harmonics related to it. In order to evaluate them, these harmonics are numbered and named in increasing order of their frequencies. Generally speaking, it can be said that the higher the frequency, the lower the intensity; however the intensity of harmonics is also altered by a range of factors including the geometry of the sound body and resonance chamber, and the material of which it is made. This range of conditions is responsible for the intensity of certain harmonics and the attenuation of others. As such, the range of combinations gives rise to a vast amount of timbre which in turn gives sounds their personalities.

The real sound emitted by an instrument has four properties that are related to the development of the emission over time:

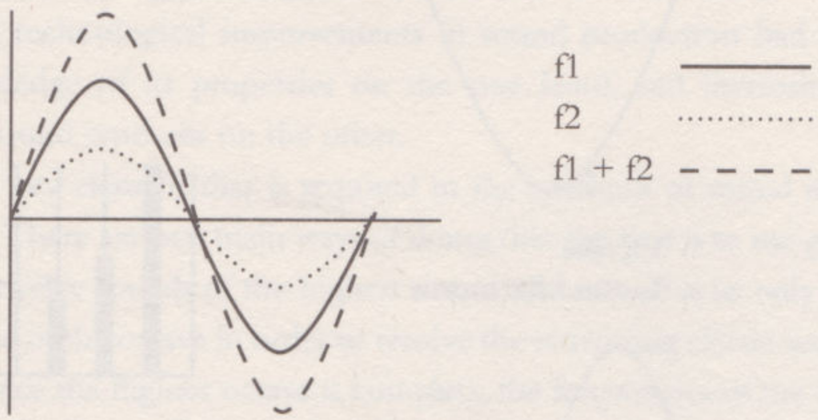
- Attack – the time which elapses from when the instrument is played and the point at which the sound reaches its maximum amplitude.
- Decay – the time which elapses between the point of maximum amplitude and the point at which the emission begins to stabilise.
- Sustain – the time for which the sound maintains its amplitude while the emission continues.
- Release – the time taken for the amplitude to drop when the emission is stopped.



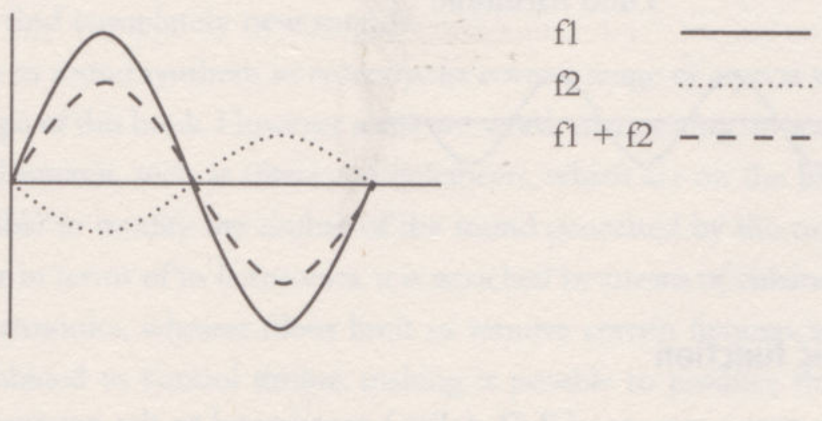
Stages in the emission of a sound with constant frequency.

Superimposition of waves

When representing the wave of a real sound we see a curve that is the superimposition of a number of different waves: the fundamental and its harmonics. Let us examine a simple case of the superimposition of waves. Consider two sounds with the same frequency but with different amplitude. If their phases coincide, this results in the amplification of the sound:

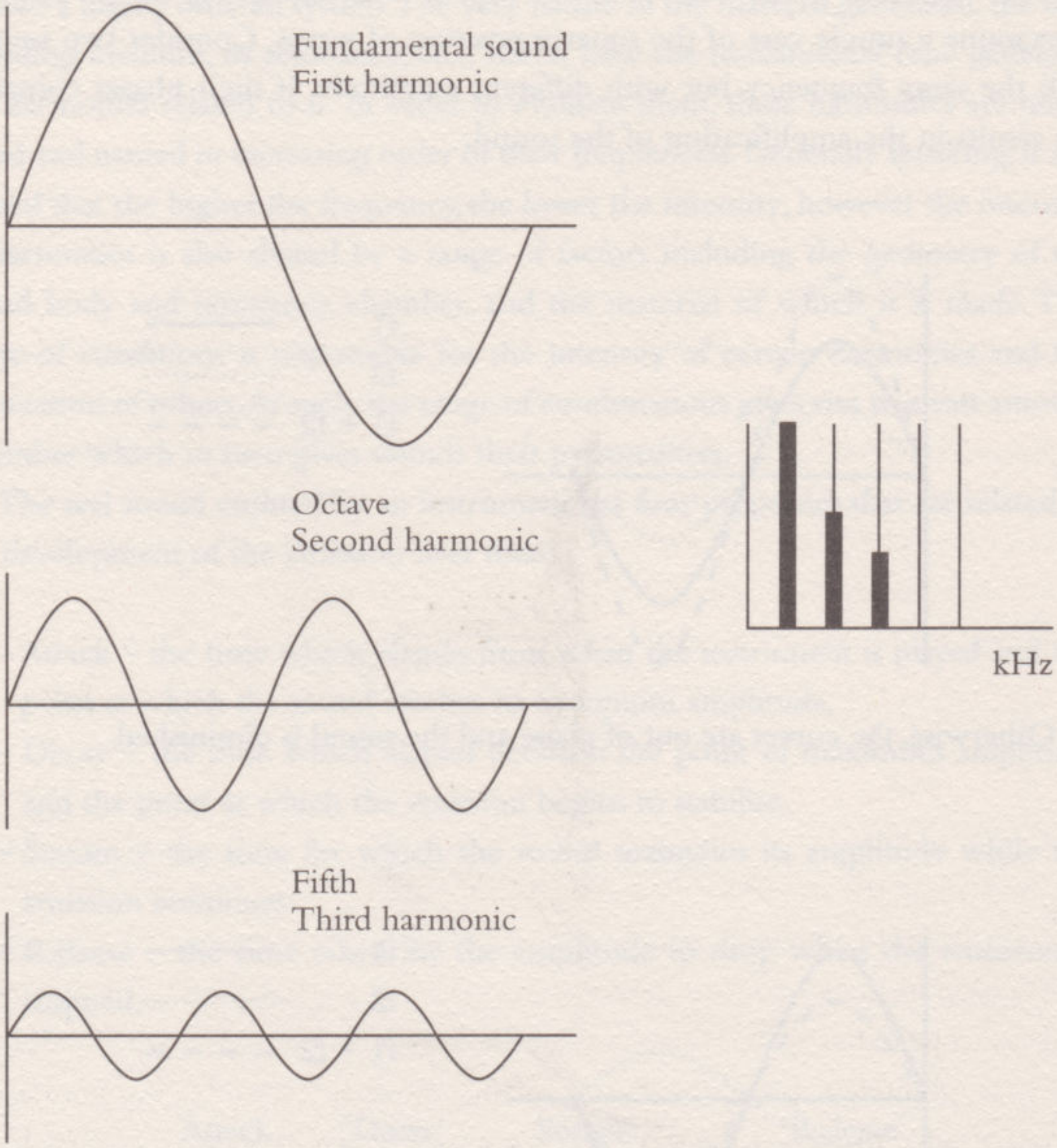


Otherwise, the curves are out of phase and the sound is diminished.



How is this peculiarity manifested in practice? Without venturing too far, the answer can be found in concert halls: the volume that can be reached by a large choir or an orchestra of strings is notably higher than that which can be reached by a vocal group of four or eight singers, or a string quartet, respectively. For more complex cases, such as the real sound produced by a musical instrument, we will obtain a non-sinusoidal graph due to the simultaneous presence of a number of

partial waves. Thanks to the Fourier principle, since these are periodical waves, it is possible to determine the frequency of each of them.



The harmonic function

Harmonics that are a power of 2 (2, 4, 8…) correspond to the octaves of the fundamental sound and their presence in the range reinforces the intensity of this sound. Harmonics that are multiples of 3 (3, 6, 12…) correspond to an octave series of fifths; their presence creates a nasal timbre. The harmonics 5, 10, 20… correspond to the third of the fundamental (and its octaves) and provide the sound with warmth. Finally, harmonics that correspond to dissonant intervals provide the sound with roughness.

The synthesis of sounds

The first attempts at creating electrical organs date back more than one hundred years. The pioneers were the American Thaddeus Cahill, who invented the Telharmonium in 1900; the Russian Léon Theremin, who invented the instrument that bears his name in 1924; and the Frenchman Maurice Martenot who devised the Martenot waves in 1928. Even if these isolated events marked the start of a new direction in terms of technology, the rise of the development of instruments to create synthesised sounds gathered pace at the end of World War II. During the previous century, technological improvements in sound production had given an in-depth knowledge of its properties on the one hand, and increasingly efficient progress in sound synthesis on the other.

The first key element that is required in the synthesis of sound is precisely its 'generation'. There are two main ways of doing this: the first is to use generators for each of the twelve sounds of the highest octave; the second is to only generate the highest sound of the octave in order to resolve the remaining eleven semitones electronically. Once the highest octave is complete, the frequencies of the other octaves are obtained by means of electronic frequency dividers that successively determine the octave of each sound using the simple ratio 2:1.

Once the base sound has been created, we then proceed to alter the different parameters to allow us to obtain the desired sound. This is a key point for sound designers, since the search for increasingly real synthetic sounds runs in parallel to the quest to find completely new sounds.

Advances in sound synthesis in recent years cover a range of aspects which are beyond the scope of this book. However some are worth mentioning on account of their mathematical interest, such as filters and enhancers, which act on the harmonics and make it possible to modify the timbre of the sound generated by the oscillator. If the sound is poor in terms of its harmonics, it is enriched by means of enhancers that produce even harmonics, whereas filters limit or remove certain frequencies. In general they are combined to control timbre, making it possible to produce the sound of a trumpet, a violin, or any other instrument. The most commonly used filters are:

- Low pass filters, which attenuate high frequencies.
- High pass filters, which do the same to low frequencies.
- Band pass filters, which attenuate high and low frequencies, allowing the central ones to pass through.
- Band-stop filter, which attenuates the central frequencies.

Digital audio

All the sounds we hear in everyday life reach us in the form of physical waves that are transmitted through air, water or any other medium through which they can pass. Since the invention of the phonograph by T.A. Edison in 1877, various analogue methods have been developed for the storage and reproduction of sound.

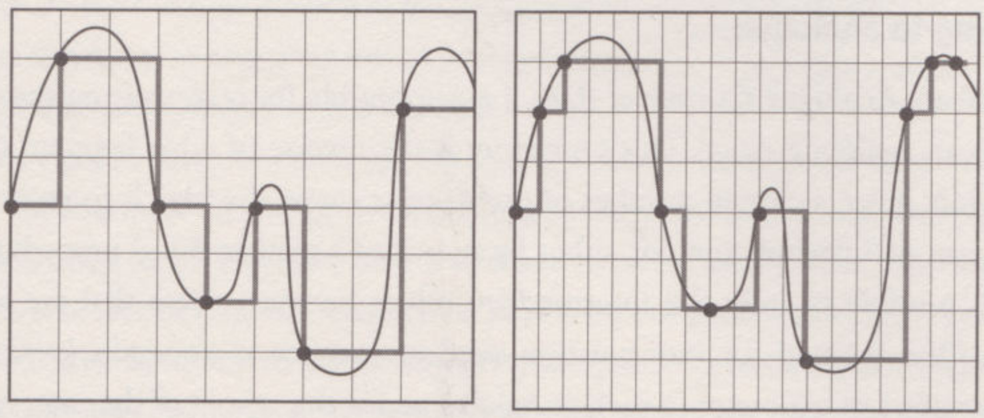
Analogue audio systems require the sound to be 'translated' by means of a transducer (e.g. a microphone) into a series of electrical pulses. These pulses are eventually recorded and reproduced and can be converted into physical waves by means of another transducer, such as a speaker.

MARY HAD A LITTLE LAMB OR *AU CLAIRE DE LA LUNE?*

Until 2008, the oldest record of the human voice was that of Thomas Alva Edison who, on 21 November 1877 recited the poem *Mary had a Little Lamb* in order to test his recently invented phonograph. Some days later, he gave the first public demonstration of his invention and, a year later, patented it and presented it to the French Academy of Sciences. The invention was so amazing that the scientists who were present believed it to be a fraud and thought there was a ventriloquist in the room. The vibrations produced by the sound were literally engraved on a layer of tin foil (although the material would later be replaced by wax) spread over the surface of a cylinder turning on its axis. The acoustic record in the form of a spiral track was then transformed into sound at a later point. During the first years of its life, the phonograph was used as a Dictaphone for companies and governmental institutions; in fact, Edison never thought that the fundamental use of his invention would be for capturing and reproducing musical performances, and at the start he resisted such a usage to the point of prohibiting it. However, the musical cylinder spread throughout the world, with flat discs being introduced towards 1890. However, twenty years prior to Edison's first recording, the Frenchman Édouard-Léon Scott had invented the phonautograph which made it possible to record the first vibrations, despite not being able to play them back. The records of the phonautograph are stored in the US Library of Congress. In 2008, a group of researchers managed to process one of these records, dated to 1860. From the mists of the static emerged the easily recognisable notes of the French tune *Au Claire de la Lune*, the oldest acoustic record in history.

Digitisation

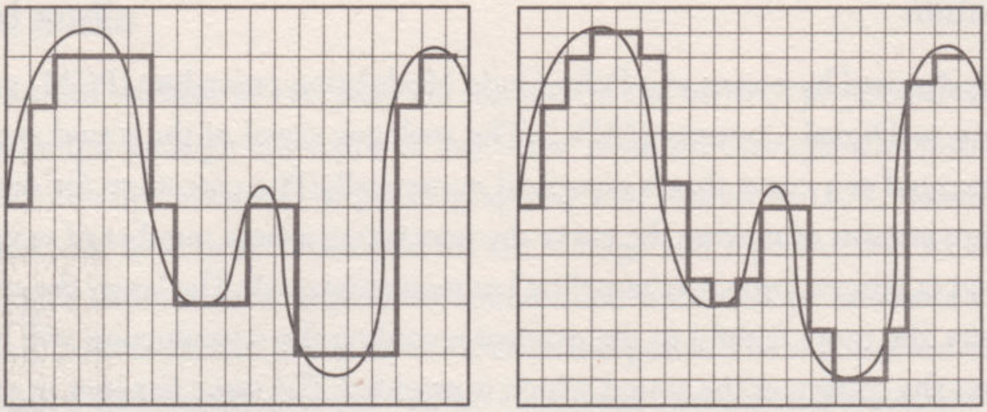
Sound is digitised by means of a Pulse-Code Modulation procedure (PCM) with an Analogue-to-Digital Converter (ADC). The analogue signal of the sound wave can be represented as a curve that is described numerically. The procedure for digitising audio thus consists of making the curve discrete: taking a large number of samples of the sound (a process known as 'sampling') at regular intervals. The larger the number of samples, the more faithful to the original recording the reproduction will be, and the better the quality of the sound which is recorded. The same happens in cinema: the greater the number of frames per second recorded in the filming, the more detail it will be possible to obtain in the movement. An equivalent example from a graphical point of view would be that the more points of a curve we know, the closer our approximation of the curve can be.



The more samples (vertical lines) that are taken of the sound, the better the approximation made by the lines of the grid and the graphy will be more faithful to the original curve.

The Nyquist-Shannon sampling theory states that, under certain conditions, an analogue signal with a maximum frequency of M can be reconstructed if the sampling rate is greater than 2M samples per second. Given that the maximum frequency we are expected to hear is 20,000 Hz (the limit of human hearing), the sampling rate for CD audio is 44,100 samples per second, just over double.

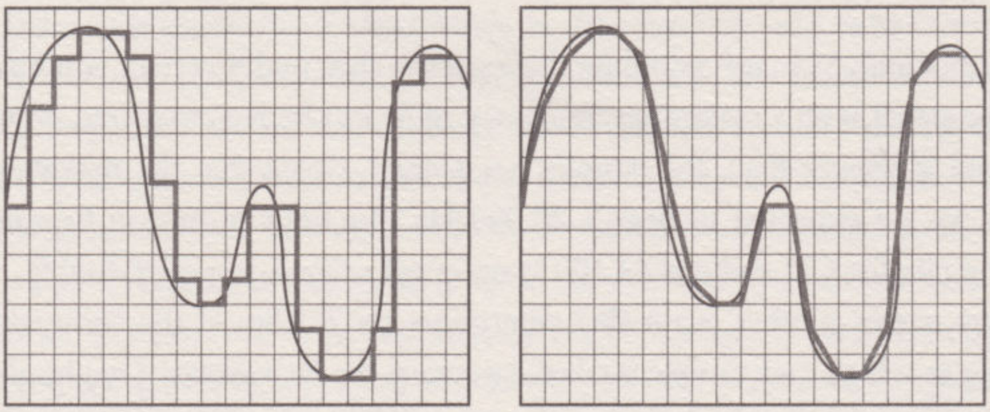
There is also another factor that determines the fidelity of the conversion: the precision to which each sample is taken, known as the *bit resolution*. Graphically, each sample has a certain height that must be measured. This measurement can be to various degrees of precision: the more bits that are available to do so, the more the space to be measured can be subdivided and the greater the precision of each sample.



Again, the more detail (more horizontal lines) that are present in each sample, the more faithful to the original curve the approximation of the lines of the grid will be.

Returning to analogue

A Digital-to-Analogue Converter (DAC) is responsible for reconverting the digital audio to an analogue signal. This procedure is the reverse of what happens during digitisation: given a certain number of points on a curve, the idea is to reconstruct it by means of 'interpolation', or rather by means of a mathematical procedure that makes it possible to infer the intermediate values between those that are already known. One method for interpolation used in practice is *zero-order hold*, which simply consists of maintaining each sample value for the whole of the interval. The other method is *first-order hold*, which works using a linear approximation of the curve between each pair of known values.



Left, interpolation using the zero-order hold method. The values of each interval between two known values are assumed to be identical to the one on the left; the line is kept horizontal. Right, interpolation using the first-order hold model. The values of each interval between two known points are determined by the gradient joining these points.

BEETHOVEN, BAYREUTH, NAZISM AND THE BIRTH OF THE CD

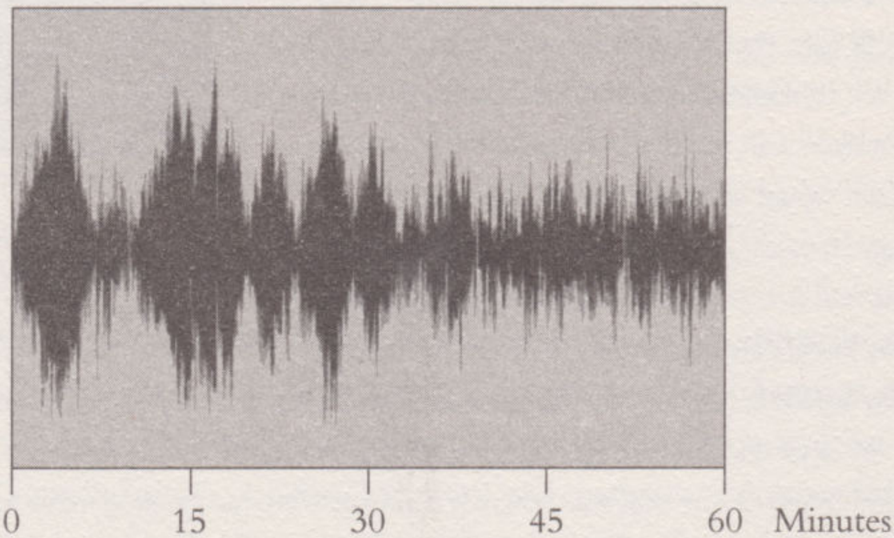
Commemorative stamp for the death of Wilhelm Furtwängler who died in 1954.

In the 1980s, CD technology was already sufficiently developed so as to be commercially viable. At the time, the Dutch company Philips and the Japanese company Sony were leaders in the audio and electronics market. Sony presented a prototype disc with a diameter of 120 mm and a capacity of 74 minutes. For its part, Phillips developed a 115 mm prototype which could hold 60 minutes of music. The Sony prototype won out and thus became the standard which would dominate the technology for more than 30 years. A curious aspect of the CD as such was that it was to have a capacity of 74 minutes. Why this apparently arbitrary number? At the time, Sony argued in favour of a prototype that made it possible to store some of the great works of universal music on a single disc, including Beethoven's *Ninth symphony*, according to Norio Ohga, former chairman. There is a definite consensus among music lovers that the reference recording for this monumental work is the one performed in 1951 under the direction of the German conductor Wilhelm Furtwängler for the reopening of the Festival of Bayreuth after World War II. This festival, held in the German city of that name, had been home to annual productions of the operas of Richard Wagner since 1876, and due to his descendants' sympathies for the Nazi cause, had become a symbol of an aggressive and bellicose Pan-Germanism in the years prior to the conflict. The reopening, which was long awaited by all of Germany, was understood as a turning point for a nation devastated and overwhelmed with guilt. Through the *Ninth Symphony*, an expansive and universal work, crowned with the immortal *Ode to Joy*, it reclaimed its place in the concord of civilised nations and turned the page on its tragic recent past. As such, the moment was a historical one and Furtwängler and his musicians rose to the occasion, with a performance of such intensity and emotional impact that when it had ended, the public were stunned and remained in silence for a number of seconds before erupting into a wave of claps and cheers that went on for almost one hour. Naturally the sound techniques used to record the event omitted this last part, and Furtwängler's *Ninth* performed in Bayreuth was immortalised in a recording of 74 minutes.

Compression

'Raw' audio

A sound wave is graphically represented on the time axis. To represent this sound wave on paper, we need a sheet that is uniformly proportional to the duration of the sound in question:



Put another way: the sound has a constant rate of information. Analogue sound systems work with a constant rate of information: It suffices to note that the speed of a turntable or a tape reel remains constant during both recording and reproduction. Audio digitisation is also processed at a constant rate, generating a 'raw' audio file. The CD-quality 'raw' audio file contains a large quantity of information, meaning that it requires large amounts of space to be stored, or high bandwidth for its transmission. This makes such files natural candidates for a compression process.

Data compression

Data compression is a process that makes it possible to reduce the numbers of bits required to encode digital information. Digital audio is compressed using formats such as MP3, FLAC and Vorbis, both in order to store it in smaller files and to be able to increase the speed of their transmission. In both cases, the audio must be decompressed before being reproduced.

Compression can be carried out by means of a number of different encoding algorithms. There are two basic types: those that cause the loss of information and

those that do not. Compression with the loss of information leads to irrecoverable degradation of sound quality, whereas if there is no such loss, the sound quality is not degraded, meaning it is possible to restore the audio to its original state. The most common compression methods (such as ZIP, RAR, ARJ) make use of algorithms that do not result in degradation of the quality of files. If they did, any text that was compressed would lose some letters or words during the process of compression and decompression.

It is also important to take the processing speed of each algorithm into account. A more complex algorithm may eventually achieve better compression, however if it requires too much processing time for compression and decompression, it is likely that it will not be of much use for real-time streaming.

Which is the best format to use? How much should we compress? Each instance strikes a balance between how much of the original quality is to be preserved and how much the storage space and transmission time is to be reduced. For professional use, it is best to conserve quality. For other uses, such as everyday listening, streaming or telephone communications, compressed files are preferred.

Methods of compression

One of the main methods of compression involves the identification of patterns and repetitions. How can each of the following binary sequences be compressed.

a) 11111111111111111111111111111111...
b) 10110111011110111110111111011111...
c) 11010110001011010000101001110010...

To approach the idea of compression, we can think of the instructions that must be transmitted to another person so they can reproduce them.

The first sequence (a) is easily transmitted since it suffices to indicate the instruction "always write a 1".

The second sequence (b) is somewhat harder: "each time write an extra 1, separating each group with a 0".

The final sequence (c) is the most complex: its irregularity leaves little room for providing instructions which offer savings over transmitting each of the numbers one by one.

Pattern recognition is fundamental to the compression of text and images. However, the information contained in an audio file is relatively chaotic, meaning that these methods do not result in great increases in compression.

Consequently, audio compression makes use of other strategies, such as psychoacoustic ones. One such strategy consists in the identification and elimination of 'perceptually irrelevant' information (a controversial expression to say the least), or rather sounds which, in principle, are not meant to be heard by the listener, or which are difficult to hear.

Another is 'noise shaping', which attempts to shift noise to frequency regions which weaken it in the perception of the listener, who will thus receive a (perceptually) cleaner sound.

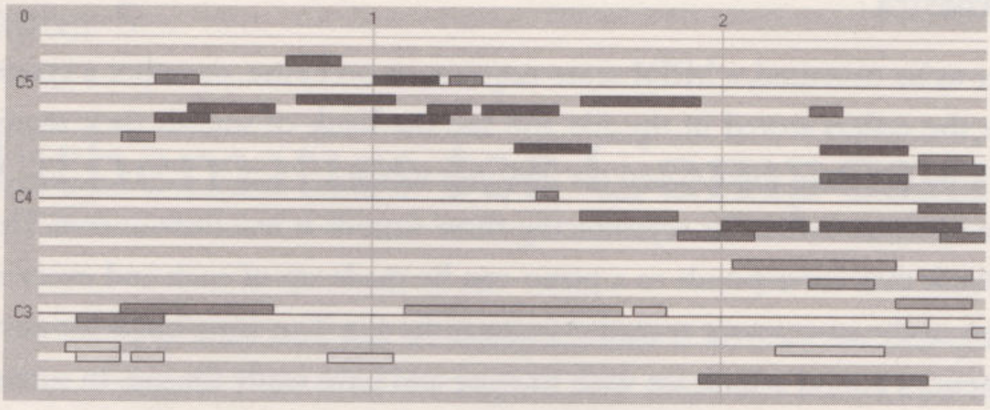
And of course, it is always possible to make use of a reduction both in the sample and bit rates.

MIDI

MIDI (Musical Instrument Digital Interface) is an instruction protocol created in 1982 to allow certain devices, such as computers and electronic keyboards, to communicate and synchronise.

MIDI instruction can be stored in files that can be executed at any time and which, since they only contain instructions, are much smaller than audio files. A MIDI file works as a digital score. It is composed of a series of events and orders which are emitted throughout the course of time. These events are able to control a large number of sound variables such as pitch, intensity, vibration and stereo panning.

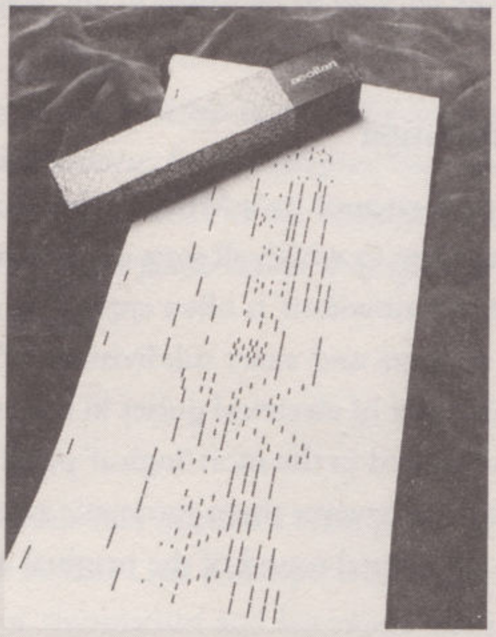
Typical instructions for a file can indicate, for example, begin a piano sound *C* with a certain intensity; stop this at point 1 and start a *D* sound with half the intensity of the *C*, etc. This simplicity makes MIDI files a highly versatile tool for musical composition. A pianist can simply sit down at a MIDI keyboard and all their actions will be recorded in a file, which can then be easily modified or adapted to meet their requirements.



An example of a 'digital score'. In the grid, time is represented on the horizontal axis. Each rectangle indicates a lapse in which each bar marks the multiples of a space or line from a conventional score.

THE MIDI DEVICES OF THE PAST

There have been several attempts to mechanise instruments. For example, string instruments were equipped with levers, while wind instruments with valves and numerous tubes and these devices paved the way for other musical machines. Of the many automation systems that have been explored, the paper roll for pianolas and barrel organs are representative of one of the first ways of recording information about sequences of sounds. The system works using rolls of paper with lengthwise perforations and cuts so that the roll represents a score for the piece of music. The time that elapses between two consecutive sounds is determined by the distribution of the holes along the length of the strip of paper, while the note to be played is defined by the position of the hole with respect to a line perpendicular to the movement. If there is a hole, a sound is played (let's call this a 'one'). If there is no hole, no sound is played (a 'zero'). The roll is a binary code and represents the first means for the automated reproduction of a piece of music.



A roll from a pianola.

The orchestra

In order to be executed, a MIDI score requires an orchestra, or rather a system that receives the orders stored in the file and has the bank of sounds they require (or knows how to create them based on the ones it has).

One obvious source for reproducing these sounds is real instruments. Thus the sound bank that corresponds to a piano can be made up of each and every one of the notes of the instrument recorded in an audio file. The same holds for any other instrument or sound which we wish to store and have ready for use. In some cases, not all the notes are recorded but, for instance, one in every three instead, recreating the other notes based on those which are known. However, at the other extreme, it is highly common to record more than one version of each note with different intensities, played in different ways, with and without the pedal, etc.

Another source of sounds is 'synthesis', artificial sounds created from scratch or by the transformation of other sounds. In general, MIDI synthesisers use the equal temperament tuning system (as explained in the first chapter) although they are versatile enough to be tuned in any way that is desired.

Quantisation

Any performance on a MIDI keyboard that is recorded in real time will generate a digital score to which all sorts of corrections, adaptations and improvements can be made. 'Quantisation' is often applied to this recording, setting the performance to a grid of notes and exact subdivisions of notes. The process is highly similar to the quantisation of electrical pulses in the audio digitisation process: each instruction is approximated to the most 'logical' point at which it is assumed the musician wished to play it. However these automatic corrections are based on criteria that can easily alter the natural sound of the original musical.

Chapter 5

Mathematics for Composition

The artist must lead an orderly life. Here is an exact timetable of my daily routine: I rise at 07:18. Inspired from 10:23 to 11:47. Lunch at 12:11; clear the table at 12:14. A healthy ride on horse-back round my estate from 13:19 to 14:53. More inspiration from 15:12 to 16:07. Various activities (fencing, meditation, immobility, visits, contemplation, dexterity, swimming, etc.) from 16:21 to 18:47. Dinner is served from 19:16 to 19:20. Symphonic readings out loud from 20:09 until 21:59. I normally go to bed at 22:37. I awake with a start once a week at 03:19 (Tuesday).

Erik Satie

Thus far, we have seen how mathematics makes it possible to describe and express different properties of music, its nature and discourse. In this chapter, the equation changes. It will now be mathematics that takes the lead as we embarks upon an exploration of the limits of tonality with the avant garde music that sprung up at the start of the last century.

Full equality: twelve-tone music

Towards the start of the 20th century, tonal music had reached a crisis point. In their quest for the limits of expression, composers such as Liszt and especially Wagner and Strauss had taken the harmonic principles of chromatism and harmonic ambiguity almost to the limit at which tonality was lost. 'Atonal' music, or rather music that lacked a central tone, arose as part of this process. One of the composers to experiment with this school of music was Arnold Schönberg (1874–1951). As part of these experiments, at the start of the 1920s the Austrian musician developed the compositional technique referred to as the 'twelve-tone technique', which attracted the attention of other composers, such as such as Alban Berg and Anton von Webern who formed part of the so-called Second Viennese School.

What is the twelve-tone technique?

The term 'twelve-tone technique', also referred to as 'dodecaphony', refers to the different sounds in the Western musical system (*dodeca* means 'twelve' in Greek). The sounds correspond to the seven white and five black keys of the piano. The use of twelve tones requires that two important points must be taken into account.

- The twelve-tone technique ends up finally unifying sounds that had previously retained separate identities, such as *F sharp* and *G flat*, and uses one or another indeterminately, treating them as equals.
- The references made to each of the twelve sounds include all those of their class: Thus a *C* does not to refer to any *C* in particular but to all the *C* notes of the various octaves, each functioning as a representative of all those in its class. Hence there are only 'twelve' sounds.

The twelve-tone technique maintains the idea of an atonal music, distancing itself from strong hierarchical attraction towards a single note (the tonic) that stands out above the others. The technique developed a method of avoiding the preponderance of certain notes over others by preassigning each the same relative value and arranging the music so that all notes would appear approximately the same number of times in a composition.

RULING OUT A THIRTEEN-TONE TECHNIQUE

The fact that Schönberg, the father of the twelve-tone composition system, suffered from triskaidekaphobia, the superstitious fear of the number 13, may strike the reader as odd. The origin of this phobia is uncertain, although we know it has been around for an extremely long time, affecting the Vikings and the Christian tradition, the latter associating it with Judas, who occupied the thirteenth place at the table of the Last Supper. In ancient Persia, the number is associated with chaos. Fear of the number 13 has been taken to incredible extremes. Thus many cities whose streets are numbered do not have a 13th street and many buildings are also built without a thirteenth floor. In Formula 1 too, no car is identified by the number. The American actor, Stan Laurel, from the famous duo Laurel and Hardy, was actually known as Stan Jefferson (13 letters), and changed his surname on account of this very superstition. Some musicians have also

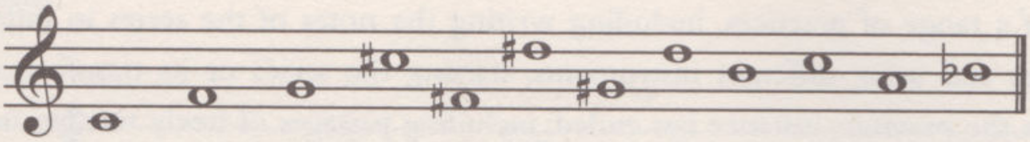
Series

In order to achieve this objective, the method imposes a series of compositional rules. For instance, to avoid the listener focussing on some notes at the expense of others, the compositions must complete cycles using the twelve available notes. When one note has been used, it may only be used again once the cycle of twelve notes is complete.

The notes of the cycles were not presented in a random manner but instead each composition was structured based on a 'series', or rather a precise ordering of the twelve available notes.

However, the series is not just an arrangement that serves a statistical purpose, but it receives a motif based traditional treatment. In this respect, the twelve-tone technique considers itself as an heir to the Western musical tradition.

The following series appears in Schönberg's *Suite* op. 25, one of the first works to make use of the twelve-tone technique.



The composer developed the series together with other connected or derived ones and in order to obtain these, the twelve-tone technique makes use of the

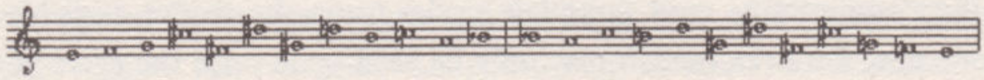
shown a certain aversion to the number. In his record *Room for Squares*, the American John Mayer recorded 14 tracks, leaving 2 seconds of silence for the thirteenth; generally speaking, this track is avoided in numbering.

Arnold Schönberg was born on 13 September 1874. He came to change the name of his opera *Moses und Aaron* to *Moses und Aron* to avoid the title having 13 letters. He was afraid of dying in a year that was a multiple of 13 and in 1950, when he turned 76 years old (7 + 6 = 13), he became depressed. He died on 13 July 1951.

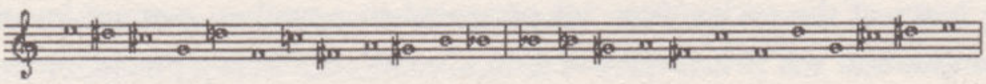
For his part, Alban Berg had become obsessed with the number 23, which he considered to be 'fatal'. However this number has a strong presence in his *Lyric Suite*: the number of bars in many of its sections is a multiple of 23, as are the metronome marks.

compositional transformations we saw in chapter 3 – inversion, retrogression and transposition.

Op. 25 Original Retrogression



Inversion Retrograde inversion



There is also a fourth transformation that some composers include in their palette: 'rotation'. If we arrange the series in a circle (connecting the last note to the first) a rotation is the result of starting the series at any of its points.

Twelve-tone notation is not as strict as the use of series would appear to suggest. While they may form the backbone of twelve-tone music, each composer has adapted them to meet their own requirements. Based on the series, a composer can make use of a range of practices, including writing the notes of the series in different octaves and using different instruments; starting the series or its transformation before the previous instance has ended; including passages of freely written music; working with derivative series made up of fragments of the generator series, etc.

HOW MANY DIFFERENT SERIES ARE THERE?

The first note of the series can be any of the twelve that are available. Once the first note has been chosen, the second can be any of the eleven remaining notes, giving a partial result of $12 \cdot 11$ series. Once the first two have been chosen, the third can be any of the ten remaining notes, giving a partial result of $12 \cdot 11 \cdot 10$ series. Continuing this reasoning, the numbers indicate an initial total of $12 \cdot 11 \cdot 10 \cdot 9 \ldots \cdot 3 \cdot 2 \cdot 1 = 479{,}001{,}600$ different series. This number is known as '12 factorial' and is written as 12!

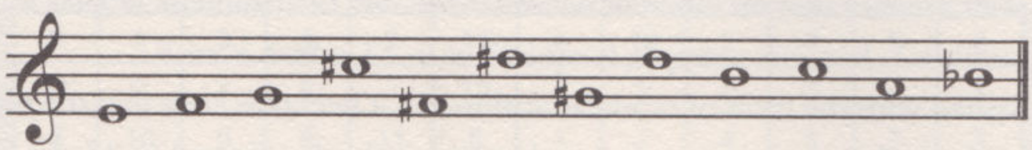
In general, for any positive integer n the factorial of n is defined as the product of all the positive integers from 1 to n. Thus $n! = n \cdot (n-1) \cdot \ldots \cdot 2 \cdot 1$.

However, in the case of twelve-tone series, the counting is a little more complex, since if we wish to know the number of series that are 'essentially' different, we must discount transpositions, inversions, retrogressions and combinations of these operations. A careful count indicates that the number of possible series is 9,985,920.

Numerical and matrix representation

Traditional stave-based scores follow the logic of diatonic music. One consequence of this is that the distance between neighbouring lines and spaces does not always represent the same musical distance: sometimes it represents two semitones (from *D* to *E*), and others just one (from *E* to *F*)… This means that twelve-tone music must be written making use of alterations. For this reason, and as is made clear in the previous examples, the inversions and retrogressions of the twelve-tone series are not wholly 'visible' on the scores, although the music is still written on them.

A series can also be represented numerically, simplifying the preparatory work for the notation of motifs based on series and connected series. When we wish to represent a series numerically, a starting note is generally taken as a reference point. In the following example, this reference note is *E*, which is assigned the value 0. The other pitches are successively numbered by semitones: *F* is 1; *F sharp* is 2; *G* is 3, etc. Each note from the series is assigned a number that indicates the class to which it belongs.



n° 0 1 3 9 2 11 4 10 7 8 5 6

Representing a series of notes numerically makes it possible to use arithmetic for calculating other series connected to it. For example, the transposition of a series is obtained by adding the same number, *k*, to each of its elements:

$T_k\,(s_1, s_2, \ldots, s_{12}) \rightarrow (s_1+k, s_2+k, \ldots, s_{12} + k)$
$T_0\,(0, 1, 3, 9, 2, 11, 4, 10, 7, 8, 5, 6) \rightarrow (0, 1, 3, 9, 2, 11, 4, 10, 7, 8, 5, 6)$
$T_1\,(0, 1, 3, 9, 2, 11, 4, 10, 7, 8, 5, 6) \rightarrow (1, 2, 4, 10, 3, 0, 5, 11, 8, 9, 6, 7)$
$T_2\,(0, 1, 3, 9, 2, 11, 4, 10, 7, 8, 5, 6) \rightarrow (2, 3, 5, 11, 4, 1, 6, 0, 9, 10, 7, 8)$
…
$T_7\,(0, 1, 3, 9, 2, 11, 4, 10, 7, 8, 5, 6) \rightarrow (7, 8, 10, 4, 9, 6, 11, 5, 2, 3, 0, 1)$
…
$T_{11}\,(0, 1, 3, 9, 2, 11, 4, 10, 7, 8, 5, 6) \rightarrow (11, 0, 2, 8, 1, 10, 3, 9, 6, 7, 4, 5)$

After 11, we return to count from 0, just like the hours of the day: seven hours after eight o'clock in the morning gives three o'clock in the afternoon. In mathe-

matics, these types of operations on reduced sets of numbers are referred to as 'modular arithmetic'. In the case of twelve-tone series, the set is made up of numbers between 0 and 11, giving a total of 12. The number of elements in the set is referred to as the modulus (in this case, 12). Thus in modulus 12 arithmetic, the number 13 is equivalent to 1 and is written as:

$$13 \equiv 1 \pmod{12}.$$

All numbers of the form $12k + 1$ are also equal to 1 where k is an integer:

$$25 \equiv 1 \pmod{12}$$
$$37 \equiv 1 \pmod{12}$$
$$49 \equiv 1 \pmod{12}$$
$$61 \equiv 1 \pmod{12}$$
$$\cdots$$

As we have already observed, the twelve-tone technique does not distinguish between similar notes on different octaves. Modulus 12 arithmetic reflects this, since the number 1, *F* in our example, is equivalent to 13, which is an *F*.

Armed with the tools of modular arithmetic, it becomes clear that the inversion of a series is the same as assigning each numerical value from 0 to 11 (i.e. each of the different notes) the difference between that number and 12. The value 1 in the series thus becomes 11; 2 becomes 10; 3 becomes 9… In the case of the series in our example:

$$I\,(s_1, s_2, \ldots, s_{12}) \rightarrow (s_1, 12\text{-}s_2, \ldots, 12\text{-}s_{12})$$
$$I\,(0, 1, 3, 9, 2, 11, 4, 10, 7, 8, 5, 6) \rightarrow (0, 11, 9, 3, 10, 1, 8, 2, 5, 4, 7, 6)$$

The retrogression is obtained by 'turning around' the numerical series, reading it from right to left:

$$R\,(s_1, s_2, \ldots, s_{12}) \rightarrow (s_{12}, s_{11}, \ldots, s_1)$$
$$R\,(0, 1, 3, 9, 2, 11, 4, 10, 7, 8, 5, 6) \rightarrow (6, 5, 8, 7, 10, 4, 11, 2, 9, 3, 1, 0)$$

The original series, with the inverse, the retrogression, the retrograde inverse and each of these 4 with their 12 possible transpositions give the composer a pallet of $4 \cdot 12 = 48$ permutations available for use in their work. (If we also consider

rotations, the number of permutations rises to $48 \cdot 12 = 576$.) These 48 forms can be represented in a 12x12 matrix, following these rules:

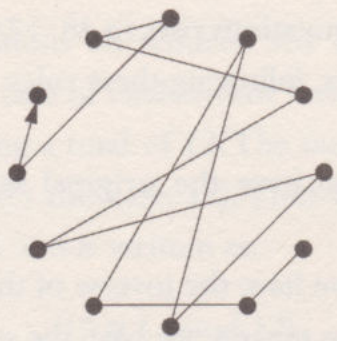
– In the first row T_0 we have the original series (marked in bold in the example).
– In the first column I_0, we have the inverse of the series (also in bold).
– In each of the remaining squares, we have the sum (modulus 12) of the numbers in the row and column headers. For example, the fifth row begins with a 10, and the fourth column with a 9, meaning that the box in which they meet should have a 7, since $10 + 9 = 19 \equiv 7$ (mod. 12).

Thus the twelve rows contain the original series with all its transpositions, and the twelve columns contain the inverse of the original series with all its transpositions. The retrogressions of these twenty-four are simply obtained by reading the matrix in the other direction: the rows from right to left and the columns from bottom to top.

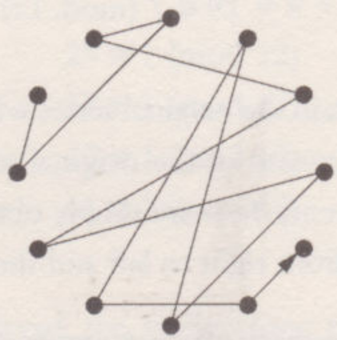
	I_0	I_1	I_3	I_9	I_2	I_{11}	I_4	I_{10}	I_7	I_8	I_5	I_6	
T_0	**0**	**1**	**3**	**9**	**2**	**11**	**4**	**10**	**7**	**8**	**5**	**6**	R_0
T_{11}	**11**	0	2	8	1	10	3	9	6	7	4	5	R_{11}
T_9	**9**	10	0	6	11	8	1	7	4	5	2	3	R_9
T_3	**3**	4	6	0	5	2	7	1	10	11	8	9	R_3
T_{10}	**10**	11	1	7	0	9	2	8	5	6	3	4	R_{10}
T_1	**1**	2	4	10	3	0	5	11	8	9	6	7	R_1
T_8	**8**	9	11	5	10	7	0	6	3	4	1	2	R_8
T_2	**2**	3	5	11	4	1	6	0	9	10	7	8	R_2
T_5	**5**	6	8	2	7	4	9	3	0	1	10	11	R_5
T_4	**4**	5	7	1	6	3	8	2	11	0	9	10	R_4
T_7	**7**	8	10	4	9	6	11	5	2	3	0	1	R_7
T_6	**6**	7	9	3	8	5	10	4	1	2	11	0	R_6
	RI_0	RI_1	RI_3	RI_9	RI_2	RI_{11}	RI_4	RI_{10}	RI_7	RI_8	RI_5	RI_6	

Circular representation

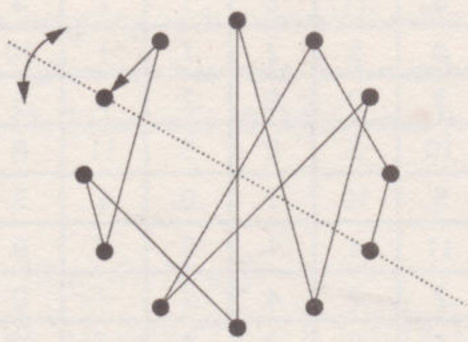
The circular representation of a series is especially useful for studying the twelve-tone technique. For example, the circular representation of the series from Schönberg's op.25, which we saw above, is as follows:



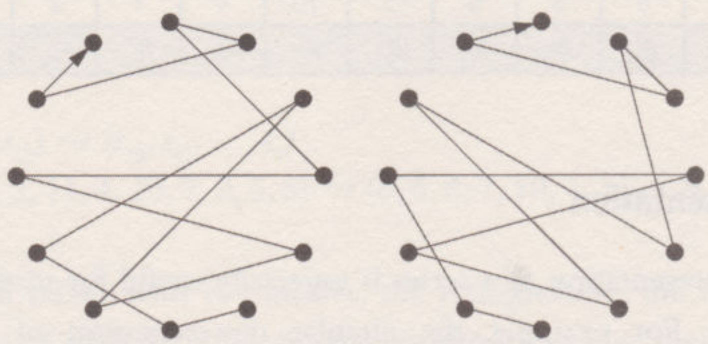
To obtain the retrogression of the series, we need only reverse the direction of the path.



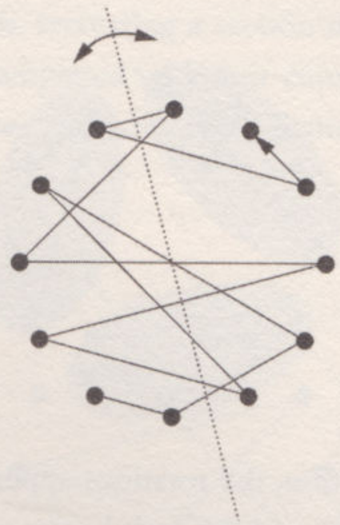
To invert it, all that is needed is to invert it with respect to the axis of symmetry passing through its reference tone:



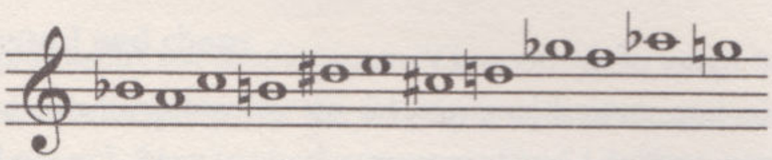
For transposition, we need to rotate it by the required number of 'hours':



The inverse of a transposition can be obtained by its reflection on the appropriate axis:

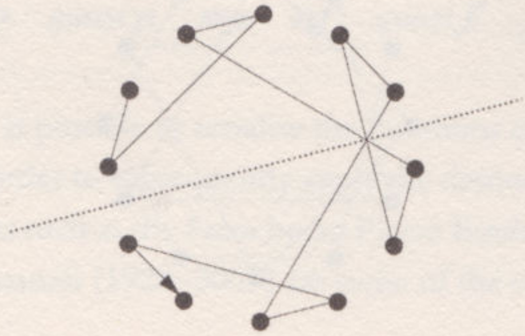


The circular representation makes it possible to appreciate the internal structure of a series better. For example, the series from Anton Webern's *String quartet* op. 28 which, as we have seen, is based on the theme BACH:

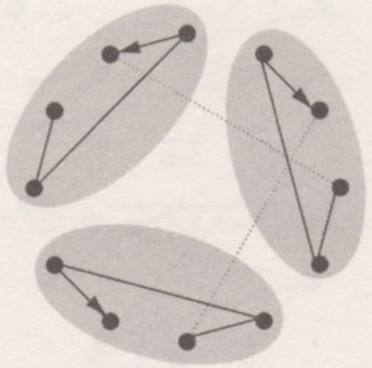


The circular representation of this same series makes it possible to clearly recognise its symmetry. In the image, the symmetry of the series has been marked by a dotted line whose halves are, reciprocally, their transposed retrogressions. This means that the series S will be equal to its retrograde inversion, transposed by three descending semitones. Or rather that this series is obtained by applying the functions we have just seen to the original series – retrogression (R), inversion (I) and transposition (T), the latter being applied three times:

$$S = T3(I(R(S)))$$



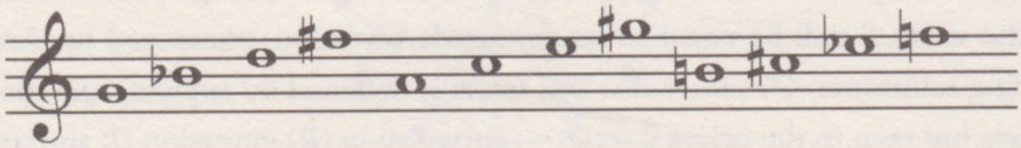
The theme BACH, which is already symmetrical in its own right, appears three times: first in its original form; then inverted and transposed; and finally transposed.



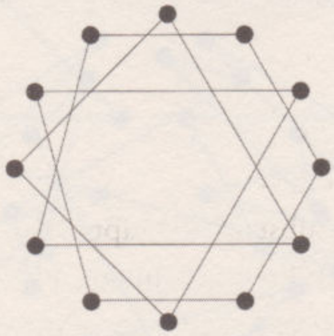
In the circular representation, the rotations appear connecting the last note to the first, completing a 'circuit', and the path begins at any of the points in the circuit.

Alban Berg

The third great figure of the Second Viennese school was Alban Berg (1885–1935). Master of an intense musical language, his use of twelve-tone techniques did not prevent him from creating a highly expressive body of work. Among his best-known pieces are the operas *Wozzeck* and *Lulu*, as well as the *Lyrical Suite* and a violin concerto. The series of this final composition is:

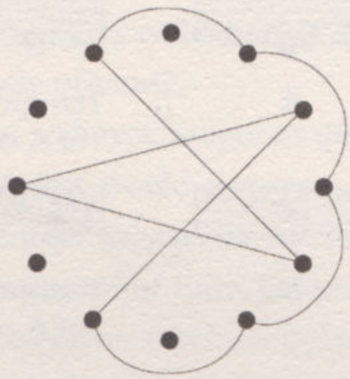


It exhibits a surprising symmetry when considered as a circuit, connecting the last note to the first:



The series in question has a tonal resonance that becomes clear when represented numerically (0, 3, 7, 11, 2, 5, 9, 1, 4, 6, 8, 10). Note that it includes a chain of four major and minor chords, recreating a section of the circle of fifths: 0-7, 7-2, 2-9 and 9-4. The circuit is completed with four consecutive tones.

The following circular diagram shows these chains of fifths, eliminating some of the intermediate elements.



Serialism, control and chaos

The twelve-tone technique paved the way for a style of musical composition that was strongly influenced by various mathematical models. The principles that applied to pitches in series were not long in being transferred to other parameters of music. The original idea was to statistically eliminate the preponderance of one pitch over another. Why not do the same with other parameters such as intensity, the durations of the notes, timbre and register? The method is essentially identical to that used for the pitches. To work with intensities, a table is created listing twelve graduations of intensity, from quadruple piano to quadruple forte. With these elements, it is possible to devise a series of intensities that are suitable for serial transformation:

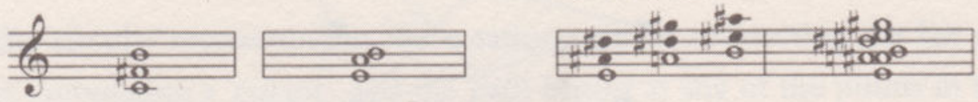
1	2	3	4	5	6	7	8	9	10	11	12
pppp	***ppp***	***pp***	***p***	*quasi* ***p***	***mp***	***mf***	*quasi* ***f***	***f***	***ff***	***fff***	***ffff***

In the same way, it is possible to serialise the durations of the notes or any other musical parameter in order to subsequently apply the desired mathematical/musical procedures. Composers such as the Frenchman Pierre Boulez (1925–) and the German Karlheinz Stockhausen (1928–2007) are some of the exponents of this school,

making systematic use of series and applying them to different musical parameters. The technique was given the name 'integral serialism'.

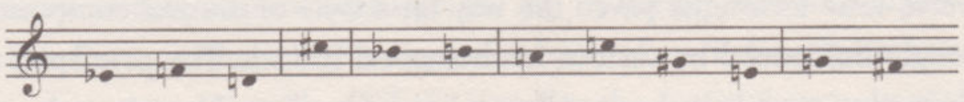
Boulez developed a procedure for 'block multiplication'. Each of the harmonic blocks A and B is a chord – a specific set of pitches. Block A is transposed by taking each of the notes of B as the lower note. The product A × B is the harmonic union of all these transpositions

A *B* Transpositions of A over B *A* × *B*



A *B* Transpositions of A over B *A* × *B*

This procedure was applied by Boulez in his work *Le Marteau sans Maître*, dividing a series into five blocks *a*, *b*, *c*, *d* and *e*, which are multiplied according to the aforementioned procedure:



a *b* *c* *d* *e*

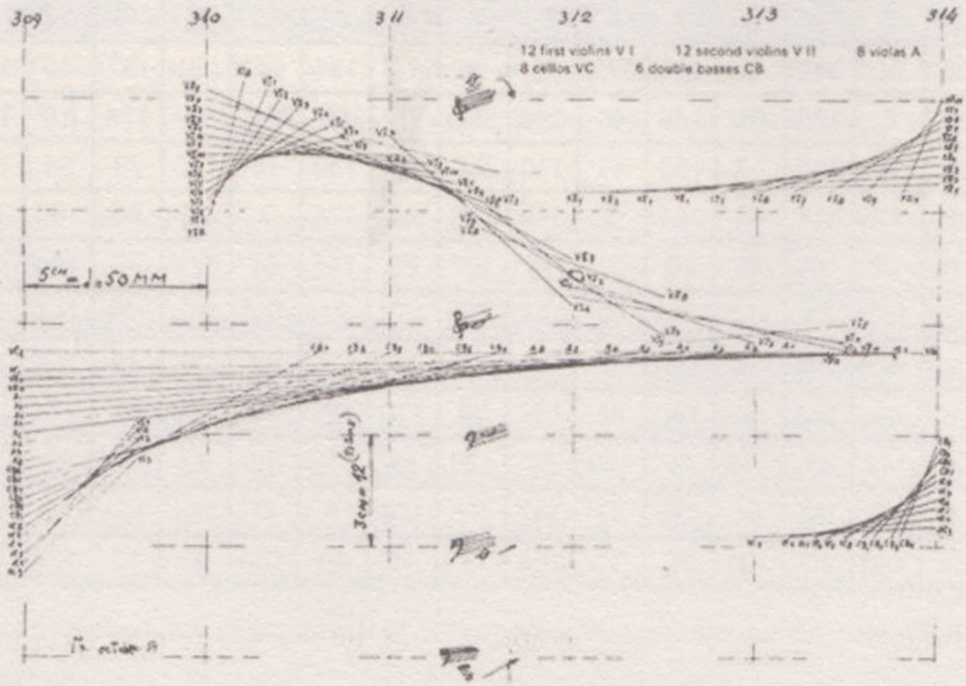
This is an interesting example of the application of a mathematical concept such as multiplication to a field in which its application did not originally appear to make sense. However, all this effort was not particularly fruitful. By restricting the compositional process to a mere abstract and isolated game, the compositions of serialism were virtually impossible to 'decode' in musical terms. Boulez himself made reference to this problem with respect to his work *Structures I*:

"I wanted to eradicate from my vocabulary absolutely every trace of the conventional, whether it concerned figures and phrases, or development and form. I then wanted to gradually win back, element after element, the various stages of the compositional process, in such a manner that a perfectly new synthesis might arise, a synthesis that would not be corrupted from the very outset by foreign bodies – stylistic reminiscences in particular."

Stochastic music

The Romanian-born Greek composer Iannis Xenakis (1922–2001) criticised serialism on the grounds that all serial planning that was independent of the various musical parameters (pitches, durations, intensities, etc.) ended up isolating these components and impeded the relationships between them. The arrangement of different series in parallel can be equated to the idea of a conceptual polyphony, with an ideal listener able to appreciate the development of each series as they did with the different melodic voices in a traditional work. However the result seemed to be more like a collection of disparate elements that did not make up a single overall body of music.

Xenakis, who was also an architect, sought to use his compositional techniques to create a structured music that would recreate a sense of consistency between aesthetics and nature. His music appears to the listener in the form of "sound clouds" that evolve over the course of time. These clouds are made up of a large number of audible particles, elements that have little individual relevance but statistically contribute to the whole. Tracing the large structural lines of the work, the distribution of these clouds is established based on a set of highly sophisticated mathematical models and tools, which draw on methods from probability, algebra, set theory and game theory.



The score for Metastasis*, by Iannis Xenakis.*

Mozart's dice game

Wolfgang Amadeus Mozart (1756–1791) and Joseph Haydn (1732–1809) are two of the best known classical composers. With an aesthetic that was highly accessible on the surface, the music of the period was governed by strict compositional rules that both these figures mastered to perfection.

The rise of the music of the classical period coincided with the Industrial Revolution, when new machines were able to automate the production processes and reproduce human work on a large scale. These changes modified the structure of social groups and economic organisation and gave rise to the idea of mass production.

Johann Philipp Kirnberger (1721–1783), composer, music theorist and student of Bach, created various temperaments which are named after him. In 1757 he published the first in a series of games which provided a systematic approach to musical composition and allowed anybody to generate their own pieces of music without requiring any musical knowledge.

Mozart and Haydn also took part in this pastime and created their own *Musikalisches Würfelspiel*, 'musical dice game'. The following tables, which are attributed to Mozart, consist of 176 pre-composed bars, numbered and arranged in the two tables. Each has 16 columns; a number must be selected from each of these, chosen at random by throwing two standard cubic dice.

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
2	96	22	141	41	105	122	11	30	70	121	26	9	112	49	109	14
3	32	6	128	63	146	46	134	81	117	39	126	56	174	18	116	83
4	69	95	158	13	153	55	110	24	66	139	15	132	73	58	145	79
5	40	17	113	85	161	2	159	100	90	176	7	34	67	160	52	170
6	148	74	163	45	80	97	36	107	25	143	64	125	76	136	1	93
7	104	157	27	167	154	68	118	91	138	71	150	29	101	162	23	151
8	152	60	171	53	99	133	21	127	16	155	57	175	43	168	89	172
9	119	84	114	50	140	86	169	94	120	88	48	166	51	115	72	111
10	98	142	42	156	75	129	62	123	65	77	19	82	137	38	149	8
11	3	87	165	61	135	47	147	33	102	4	31	164	144	59	173	78
12	54	130	10	103	28	37	106	5	35	20	108	92	12	124	44	131

	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32
1	72	6	59	25	81	41	89	13	36	5	46	79	30	95	19	66
2	56	82	42	74	14	7	26	71	76	20	64	84	8	35	47	88
3	75	39	54	1	65	43	15	80	9	34	93	48	69	58	90	21
4	40	73	16	68	29	55	2	61	22	67	49	77	57	87	33	10
5	83	3	28	53	37	17	44	70	63	85	32	96	12	23	50	91
6	18	45	62	38	4	27	52	94	11	92	24	86	51	60	78	31

The player-composer begins by throwing the dice to obtain a number between 2 and 12. This number indicates the row to be selected in the first column. For instance, if they obtain a 3 and a 5 after throwing the dice, this means that they should select the number in the first column on row 8, which is 152. This number leads them to bar 152 which will form the first in their 'piece'. Repeating this procedure for each of the remaining columns (with just one dice in the second table) gives the 32 bars.

How many pieces?

How many different pieces can be created using this game? In the first bar, there are 11 possible ways, one for each possible result when throwing the dice: from 2 to 12. For each of these there are 11 ways of completing the second bar; this gives a total of $11 \cdot 11 = 11^2 = 121$ ways of completing the first two bars.

For each of these, there are 11 ways of completing the third bar, making a total of $11^2 \cdot 11 = 11^3 = 1{,}331$ ways of completing the first three bars.

OULIPO

A combinatorial procedure similar to the one explained here was used in the 20th century by the French writer Raymond Queneau, who in 1960, together with the mathematician François Le Lionnais founded Oulipo, an acronym of *Ouvroir de Littérature Potentielle* (Workshop of Potential Literature). His work *Cent Mille Milliards de Poèmes (One Hundred Thousand Billion Poems)* consists of ten sonnets, each of whose fourteen verses can be combined with any other verse from the other sonnets. Hence for the ten possible first verses, there are 10 possible second verses, giving a total of 10^{14} possible sonnets, the number that gives the work its title.

Each bar of the *minuet* multiples the number of options by 11, and each bar of the trio corresponds to a multiplication by 6. In total, the game is able to generate $11^{16} \cdot 6^{16} = 129{,}629{,}238{,}163{,}050{,}258{,}624{,}287{,}932{,}416 \simeq 1.3 \cdot 10^{29}$ different 'pieces'. If someone wished to play all these pieces, one after the other, without taking a break and with a duration of 30 seconds per piece it would take more than 123,000 trillion years.

A curious detail is that if what was being sought was varied music, the game was 'flawed' in terms of its probabilities. When throwing two dice, the possible results range from 2 to 12, but they are not equally likely: while the number 7 comes up as a result in 6 of the combinations, the numbers 2 and 12 have just one combination for each, as can be seen in the following table:

Result	2	3	4	5	6	7	8	9	10	11	12
Combinations	1+1	1+2 2+1	1+3 2+2 3+1	1+4 2+3 3+2 4+1	1+5 2+4 3+3 4+2 5+1	1+6 2+5 3+4 4+3 5+2 6+1	2+6 3+5 4+4 5+3 6+2	3+6 4+5 5+4 6+3	4+6 5+5 6+4	5+6 6+5	6+6
Total	1	2	3	4	5	6	5	4	3	2	1

Copying the greats

A method that is often used in teaching composition is where the student is asked to write pieces in the style of great historical composers – a fugue in the style of Bach, a sonata movement in the style of Beethoven, or a prelude in the style of Debussy.

Let us take Beethoven as an example. In the process of copying his style, a student learns to master different compositional techniques that make the music 'sound like Beethoven'. But just what does the 'Beethoven style' consist of? It is possible to list a few rules, the way in which the thematic motifs work, the way in which harmony is developed, when smaller or larger melodic intervals are used, the use of dynamic silences and contrasts, etc.

Each of the musical dimensions of a style can be analysed statistically. For example, if we wish to study the properties of the thematic motifs of Beethoven's sonatas, it is possible to derive a statistic that indicates the breadth of the register they use,

or rather the interval between the lowest and highest note. A statistical study would show us how many of these motifs have a maximum span of 1 semitone, how many have a span of 2, 3, 4… (what is the minimum span used by Beethoven, or the first non-zero value in the sequence?). A similar statistic can be derived for any other parameter we wish to investigate.

However while statistical techniques allow us to recreate a general balance, they are not sensitive to context. When trying to copy a style, perhaps the distribution of the notes is not so important (it makes no difference knowing that a work has so many *C*s if when trying to copy the style we write all these *C*s together at the start). More important than knowing how many times each note has been used is knowing how the notes are linked between each other.

The 'Markov chain' is a mathematical tool that makes it possible to work towards a solution to this problem. The technique consist of statistically studying and reproducing how the different 'states' of a system follow each other. Applied to the creation of a melody, it allows us to reproduce the patterns that determine how the presence of successive notes influences the selection of the next one.

Markovian birthday

The following example uses Markov chains to generate a melody in the 'style' of the classic *Happy Birthday* tune. The table below indicates the number of times each note appears in the melody:

G	A	B	C	D	E	F	G′
8	3	2	6	2	2	2	1

It would seem that a melody that recreates the style of *Happy Birthday* should have its notes arranged in these proportions, although in reality such an approach struggles to produce a melody that is similar to the original one.

Instead of analysing how many times each note appears, Markov chains make it possible to study how one note follows another. The 26 notes of the melody are chained using 25 pairs of neighbours or transitions: The first translation is *G-G*, the second is *G-A*, etc. In total, there can be a maximum of $8 \cdot 8 = 64$ different transitions, although not all these are present in the melody.

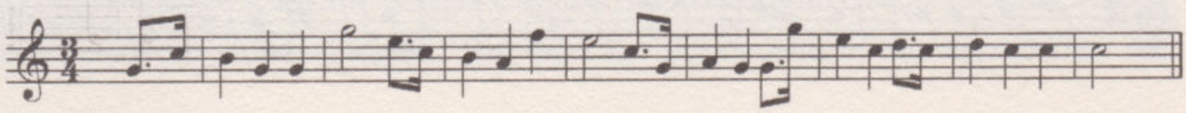
The following table indicates the number of transitions of each type.

		Next note								Total
		G′	F	E	D	C	B	A	G	
Note	G′			1						1
	F		1	1						2
	E					2				2
	D					2				2
	C				1	1	2		1	5
	B							1	1	2
	A		1						2	3
	G	1			1	1		2	3	8

Although the first note is chosen at random, it is followed by another, and this by a third, and so on, with each note being generated by a random process controlled using the transition information in the table.

Let us start the new melody on *G*, the same note on which the original melody began. Which notes can continue on from the initial *G*? The last row of the table indicates that in the melody "Happy Birthday", the note *G* has eight possible continuations: a higher *G*; a *D*; a *C*; two *A*s, and the same *G* again three times. Let us assign a number from 1 to 8 to each possible continuation and randomly select a number in this range to determine the second note of the melody: It will be the higher *G* if the result is 1, *D* if it is 2, *C* if it is 3, *A* if it is 5, and the same *G* if it is 6, 7 or 8. Let us assume that the result is 3, which means the second note of the new melody will be *C*.

The process is repeated with the 5 possible continuations of *C: D, C, B, B* and *G*. A random number from 1 to 5 will determine the third note of the new melody. Let us assume the result is 4. This means that the third note will be *B* . This process is repeated the required number of times. A melody generated using this technique is shown below:



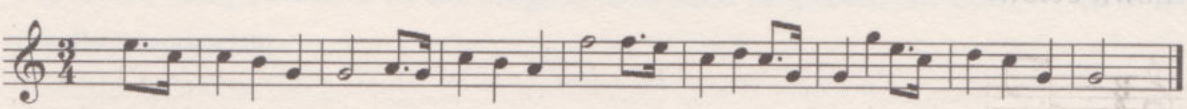
The second birthday

We have just completed an analysis using a first order Markov process, taking into account the influence of each note on the one which follows. But why not use a second order Markov process, which evaluates the influence of each pair of notes on that which follows? Let us consider the original melody once again. The first second-order transition is $G\text{-}G \Rightarrow A$. The next is $G\text{-}A \Rightarrow G$.

Although there is a universe of $64 \cdot 8 = 512$ possible second-order transitions, only a few of these are present in the melody and they are listed in the table below:

		Next note								Total
		G′	F	E	D	C	B	A	G	
Pair of notes	G′-E					1				1
	F-F			1						1
	F-E					1				1
	E-C				1	1				2
	D-C								1	1
	C-D					1				1
	C-C						1			1
	C-B							1	1	2
	C-G								1	1
	B-A		1							1
	B-G								1	1
	A-F		1							1
	A-G				1	1				2
	G-G′			1						1
	G-D					1				1
	G-C						1			1
	G-A								2	2
	G-G	1						2		3

The process for the generation of a second-order melody is the same as before, only there are now far fewer paths for 'escaping' from the original melody. The following melody is generated in this way:



These melodies seek to replicate the style of the *Happy Birthday* melody by replicating the way in which the notes succeed each other. The same technique can be used to copy other musical dimensions – the durations of notes, harmonic sequences, the registers used, orchestration, etc.

EMI

As well as attempting to replicate the style of some of the great composers, the EMI (Experiments in Musical Intelligence) programme also creates its own works.

Created by the American David Cope, EMI analyses the work of a composer and takes samples of small musical 'cells' which are then recombined to create new ones according to the style of the composer, determined by the analysis. Guided by a trained operator, EMI uses these samples to create its tables the same way as Mozart did in his *Musikalisches Würfelspiel*. In order to reconstruct these isolated fragments into a composition and hence create its own pieces, EMI makes use of various artificial intelligence techniques. Works created using EMI have been judged by human audiences: some of the listeners were delighted while others were infuriated and even disturbed by the apparent ability of machines to replicate human genius. Cope does not believe these types of reactions will persist in the future: "Ultimately, the computer is just a tool with which we extend our minds. The music our algorithms compose are just as much ours as the music created by the greatest of our personal human inspirations."

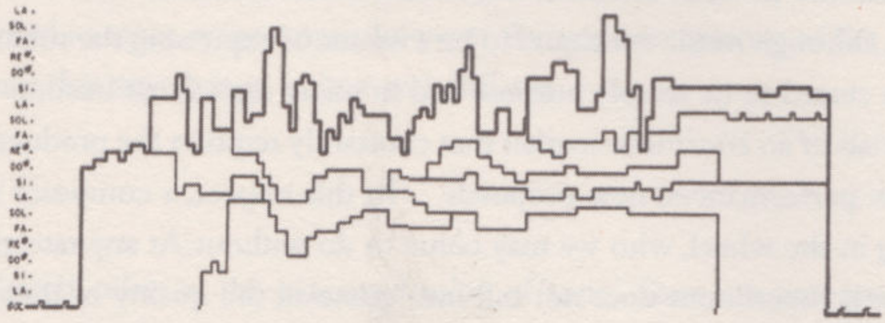
Mechanisation

Cope's program poses a profound question: is it possible to mechanise the creative process? Musical automaton were around even before Mozart's dice. In the 17th century, Athanasius Kircher created the *Arca Musarithmica*, the first instrument which followed an algorithm to design musical pieces for four voices. At the start of the 19th century, Dietrich Nikolaus Winkel (1773–1826) created the *Componium*, an automatic organ with two rolls that randomly alternated its execution. To begin to answer our question, we need to know more about composers' sources of inspiration and if there are cases in which this has given rise to processes that can be reproduced or imitated.

Inspiration

Like other types of artists, when it comes to making music, composers can draw on inspiration from the most diverse range of elements: a loved one, a historical event or figure, the work of another artist... The concertos of the *Four Seasons* by Antonio Vivaldi, the *Symphonie Fantastique*, by Hector Belioz, or the *1812 Overture* by Pyotr Tchaikovsky are some of the most famous works that can be included under the heading of 'descriptive music', that is to say music which makes explicit its connections with some kind of real event or experience. In all these cases, the source of inspiration is an environmental, historical or literary reference point that would at least mean something to the composer's contemporaries.

But inspiration does not always arise from what composers most obviously share with others. In 1939 the Brazilian Heitor Villa-Lobos (1887–1959) penned his work *New York Skyline* (subsequently revised in 1957) with its melodies suggested by the contours of the buildings of New York City, specifically by drawing their lines on squared paper.



For his part, Sir Edward Elgar (1857–1934), dedicated his celebrated *Enigma Variations* (Variations on an Original Theme for orchestra) op. 36 to "my friends pictured within". Each variation is identified by the initials – or another reference – of people close to Elgar of whom he created musical portraits. However, this is not the true enigma that gives the piece its name. There is another riddle that has yet to be deciphered: Elgar himself claims to have left a 'hidden' melody throughout the work. Like the character of a play who never appears on stage but around which the whole plot is centred, this mysterious theme is never heard in the work despite being constantly linked to the melody in musical terms. Since the publication of the work, many solutions to the enigma have been proposed, but none of them are totally convincing.

Algorithmic composition

An algorithm is a recipe, a set of instructions that indicates how a certain problem should be solved or how a task should be completed. The science of mathematics appeals to algorithms to provide instructions for carrying out basic operations. Computers carry out practically all their processes using algorithms. Although a rigorous definition (of the many existing ones) would indicate certain properties to be fulfilled by an algorithm (it must be finite, must have well-defined instructions, etc.), for our purposes, we can stick to the informal idea that it is a list of steps and/or rules to be followed to achieve a result.

Algorithmic composition models the 'conventional' process of inspiration mathematically. The composer designs an algorithm that receives certain information as an *input* and generates new information as an *output*. How can we understand an algorithm that generates music? At the end of the day, it would seem reasonable to suggest that music is a form of communication that expresses the emotions of a person and/or their vision of reality. This raises the question of why we would require a machine to write music. Is this music? What is music?

Firstly, although music continues to be a means of expressing the sublime, this has long since ceased to be its sole purpose and in many cases it has become one of the raw materials of an enormous market that constantly requires the production of new songs, new performances, new proposals... In this respect, a composer is no more than a cog in the wheel, who we may come to do without. At any rate, the fact that the person is superfluous does not call into question the quality of their work, nor place in doubt the validity of an algorithm that replaces them, but shows the standardisation of a system that previously asked little of the person and now of the algorithm.

Secondly, the design of an algorithm that is capable of 'writing' good music is a challenge that stubbornly resists computer programmers who have musical ambitions. The rules followed by music can be mathematically analysed but there is always a point beyond which the explanations appeal to vague terms such as inspiration, spirituality, sensibility and art. Is it possible to cross this boundary? Do the deepest rules of musical creation lie beyond artificial intelligence? Or will the day come when a computer programmer, with the help of modern mathematical techniques, will be able to transform it into a sort of silicon Prometheus, able to 'steal' the divine fire of inspiration, rendering it accessible to all?

Appendix I

Basic Concepts from Music Theory and Notation

This appendix provides a brief review of the basic concepts of music theory and notation in order to help you understand some of the contents of this book. Musical notation is an example of applying mathematics to an artistic discipline. Although it is perhaps not as clear as the contributions made by geometry to drawing, for example, modern musical notation includes an important set of rules and symbols that are either rooted in mathematics or open to being described using mathematical concepts. Above all, far from being a comprehensive design, musical notation is a product of a long historical evolution. Alternative, more efficient systems of notation have been proposed in more recent times, however the wide acceptance of the traditional model means that any change is slow and difficult.

Pitch

The term 'pitch' refers to the perceived value of 'tone'. Tone is a property of sound that is directly related to the frequency of the oscillation of the wave which produces it. This frequency is measured in hertz (Hz). Pitch is the property that makes it possible to distinguish between high and low sounds (higher frequencies have a higher pitch), and one note and another. The human ear is able to perceive vibrations with frequencies in the range 20–20,000 Hz. Below this range of frequencies we find so-called infrasounds and above it ultrasounds. In order to be able to order the various relative pitches of sounds, a standard referred to as the 'diapason normal *A*' or the 'reference tone' with a value of 440 Hz was defined in 1939.

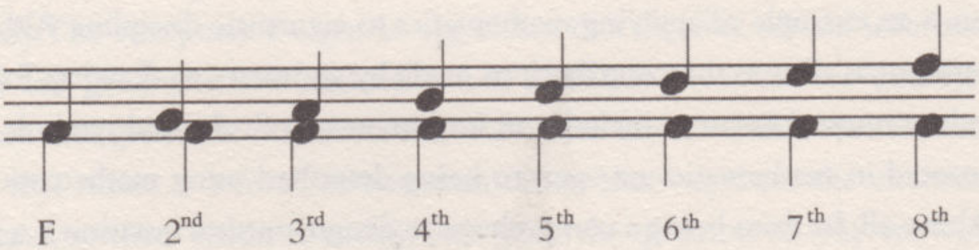
Intervals

The term 'interval' refers to the difference in pitch between two sounds perceived by a listener. Intervals are named using the ordinal number corresponding to the number of tones separating the two sounds in the scale, including the notes in

question. The definition can be better understood with some examples. If an *F* and a higher *A* are played at the same time, an interval of a 'third' is heard (*F-G-A*: three notes). Between an *A* and a higher *F* is a 'sixth' (*A-B-C-D-E-F*: six notes).

When naming two simultaneous sounds that make up an interval, the convention is to state the lower sound first, followed by the higher one. For example a second is made up of two sounds that are together on the scale: *C-D; D-E; E-F…* Likewise, examples of thirds are: *C-E, D-F, E-G, F-A, G-B…*

Thus the interval C-D is a second and the interval D-C a seventh. The complete interval between two equal notes (i.e. C-C) is an octave. An octave interval spans twelve semitones.

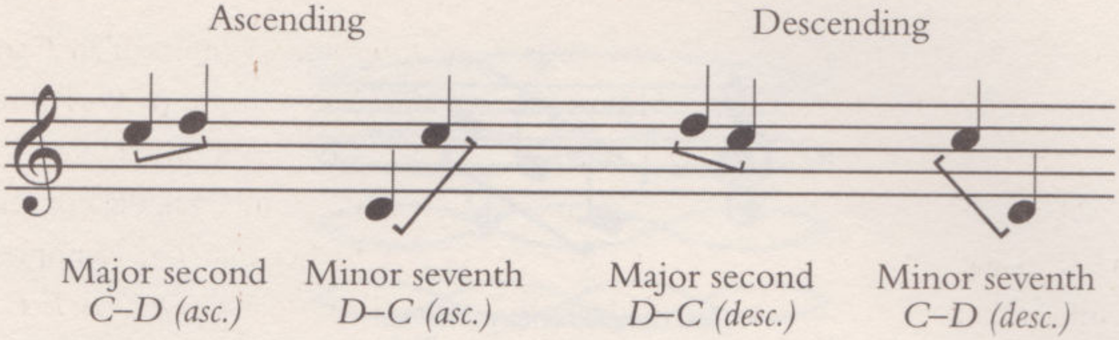


Intervals less than or equal to an octave in musical notation.

Classification of intervals

Intervals are classified as major, minor or perfect according to the number of semitones they contain. For example the two sounds in the second interval *C-D* are separated by two semitones and their full name is a 'major second'. In contrast, the notes of the interval *B-C* are separated by a single semitone and as such this is referred to as a 'minor second'. The terms major and minor can be applied to all intervals except those with five, six and seven semitones. The five semitone interval is referred to as the perfect fourth, and the seven semitone interval as the perfect fifth. A special case occurs half way through the octave: in the octave *C-C*, the *F♯* is six semitones from the lower *C* (augmented fourth) and six semitones from the higher *C* (diminished fifth).

If the sounds are played one after the other, we have a melodic interval that can either be 'ascending' or 'descending'. One of these words should be added when naming intervals. An ascending C-D interval is an ascending major second, whereas a descending C-D interval is a descending minor seventh. A descending D-C interval is a descending major second, whereas an ascending D-C interval is an ascending minor seventh. (The indication of the direction of the interval may be omitted depending on the context).



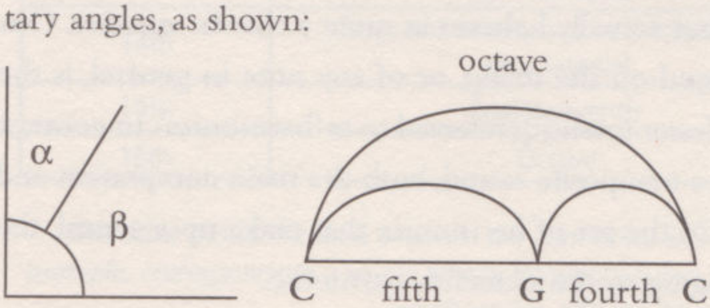
All the possible combinations of melodic intervals between two sounds.

The following table gives the measurement in semitones of the various intervals:

Interval	Measurement (semitones)
Unison	0
Minor second	1
Major second	2
Minor third	3
Major third	4
Perfect fourth	5
Augmented fourth (diminished fifth)	6
Perfect fifth	7
Minor sixth	8
Major sixth	9
Minor seventh	10
Major seventh	11
Octave	12

Inversion of intervals

The 'inversion' of an interval is another interval which, when chained to the first, completes the twelve semitones of an octave. The concept is similar to complementary angles, as shown:



The inversion of the perfect fourth (five semitones) is the perfect fifth (seven semitones): G-C *(perfect fourth)* C-G *(perfect fifth). The complement of the angle α is what is required to complete 90°, or rather the angle β.*

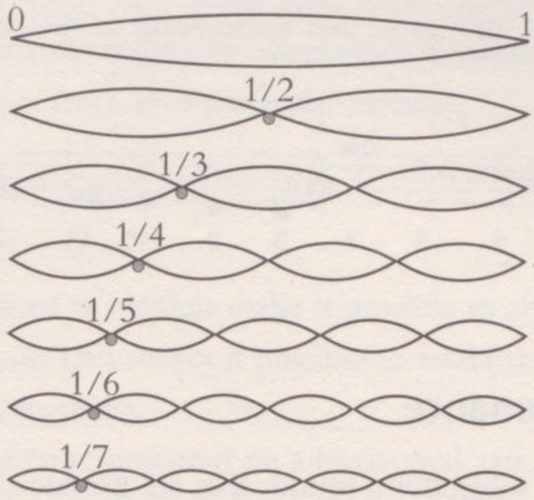
Two complementary intervals.

The following table lists intervals alongside their respective inversions:

Interval	Measurement (semitones)		Inversion
Unison	0	12	Octave
Minor second	1	11	Major seventh
Major second	2	10	Minor seventh
Minor third	3	9	Major sixth
Major third	4	8	Minor sixth
Perfect fourth	5	7	Perfect fifth
Augmented fourth (diminished fifth)	6	6	Diminished fifth (augmented fourth)
Perfect fifth	7	5	Perfect fourth
Minor sixth	8	4	Major third
Major sixth	9	3	Minor third
Minor seventh	10	2	Major second
Major seventh	11	1	Minor second
Octave	12	0	Unison

The harmonic phenomenon

When a musical instrument emits a note, in spite of the fact that this has an objective frequency *F*, the human ear does not in fact perceive it as a 'pure' tone but as the sum of an infinite number of components. A vibrating string does not move from one side to another in an orderly fashion, but actually behaves in quite a chaotic manner. Our perception of the note that is played on the string, or of any note in general, is the sum of a main tone and others of lesser intensity, referred to as 'harmonics'. In contrast to the note we perceive, which is a composite sound, both the main component and the harmonics are pure sounds. Of the set of harmonics that make up a sound, the human ear is capable of perceiving up to the sixteenth harmonic.



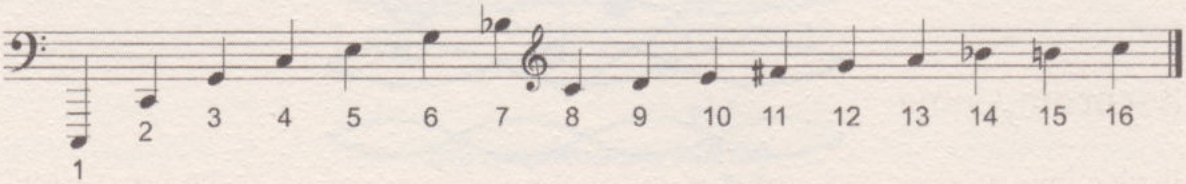
This diagram shows a string vibrating at the frequencies that correspond to the first harmonics.

In the case of an instrument that emits a *C*, the series of sixteen harmonics of the note perceived by the human ear is as follows:

No. of harmonic	Interval	Frequency	Note
1st	Fundamental frequency	33 Hz	C_1
2nd	Octave	66 Hz	C_2
3rd	Fifth	99 Hz	G_2
4th	Octave	132 Hz	C_3
5th	Major third	165 Hz	E_3
6th	Fifth	198 Hz	G_3
7th	Does not correspond to a moderated time interval	231 Hz	Bb_3
8th	Octave	264 Hz	C_4
9th	Major second	297 Hz	D_4
10th	Major third	330 Hz	E_4
11th	Does not correspond to a moderated time interval	363 Hz	$F\#_4$
12th	Perfect fifth	396 Hz	G_4
13th	Does not correspond to a moderated time interval	429 Hz	A_4
14th	Does not correspond to a moderated time interval	462 Hz	Bb_4
15th	Major seventh	495 Hz	B_4
16th	Octave	528 Hz	C_5

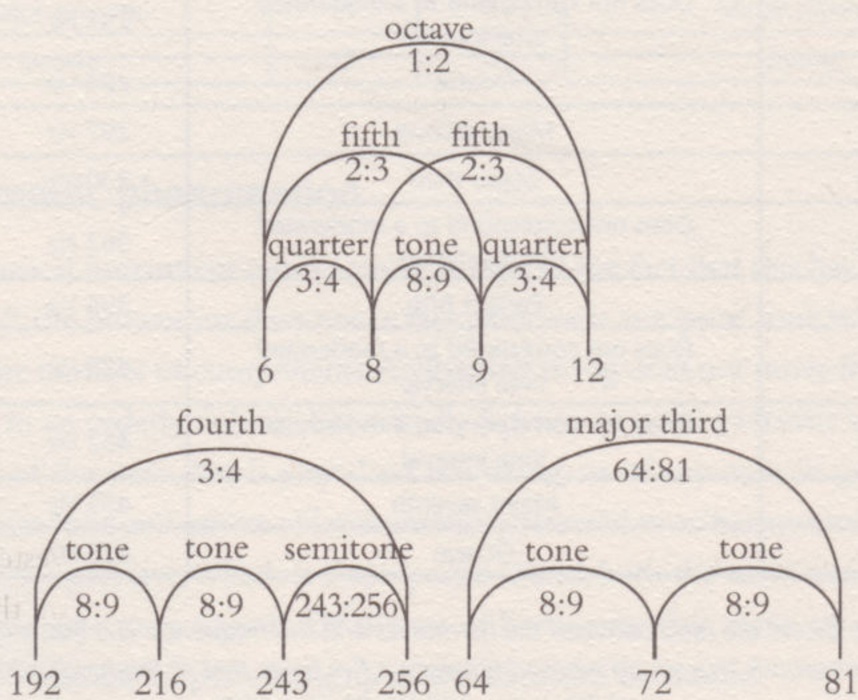
This table shows the ratio between the harmonics and the frequency. The 5th harmonic, for example, corresponds to a sound whose frequency is five times that of the fundamental frequency of 33 Hz (i.e. 165 Hz: 33 · 5 = 165).

In musical notation, the notes that correspond to the sixteen harmonics are as follows:



Consonance/dissonance

Beyond the realms of subjectivity, sounds that are produced simultaneously can be perceived as pleasant (then we speak of 'consonance') or unpleasant and filled with tensions (here we speak of 'dissonance'). In the first chapter, we noted that the Pythagorean school believed the degree of consonance between sounds was directly related to the proportion between the length of the strings which emitted two different sounds, or rather the proportion between their two frequencies. According to the Pythagoreans, intervals of an octave (produced by two strings in which the proportion of their lengths is 1:2), a fifth (where the proportion of their lengths is 2:3) and a fourth (3:4) are consonant; other intervals that are derived from the three basic ones are dissonant on account of the complex numerical ratios inherent to their sounds. The following diagrams give the most important intervals and the proportion between the frequencies of the notes of which they are composed:

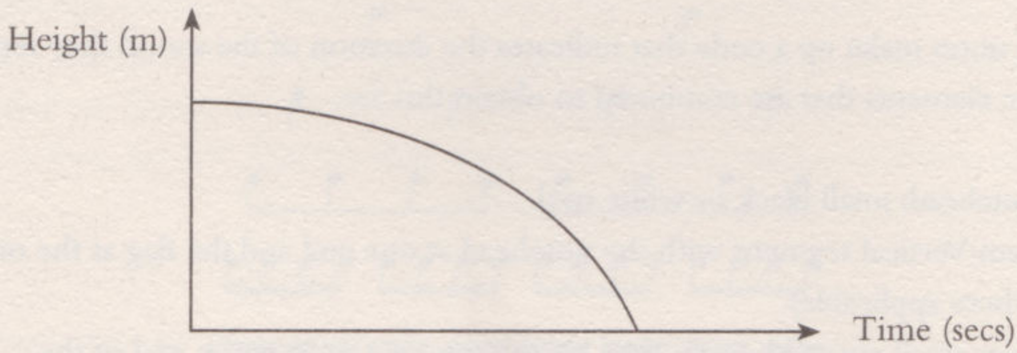


Of the many theories that have arisen since then, the concept of the 'affinity of sounds', which states that the degree of consonance between two sounds is greater the more harmonics they share, is of particular interest.

Recording time

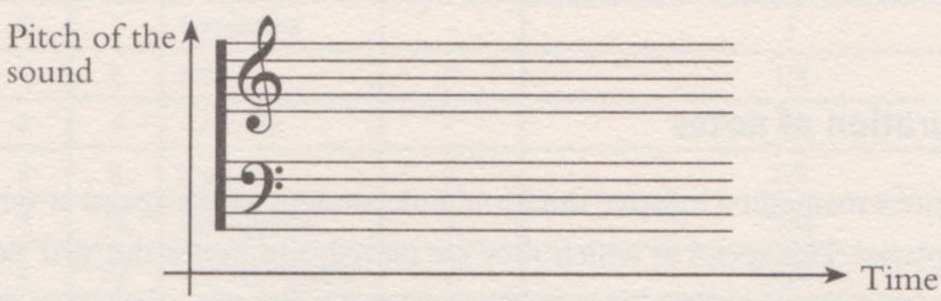
Reflections on the essence of rhythm make it possible to abstract the way various sounds and silences occur. This makes it possible to establish a more precise representation of musical phenomena.

In physics, time is often represented on a horizontal axis, from left to right. For example, a graphical representation of the position of a free falling object from the moment it begins to fall until it reaches the ground, would record its height using the vertical axis (y), with the time on the horizontal one (x). This will result in a curve with the following shape:



Graph of a free-falling body.

A similar convention has been for representing music in time.



Music is read from left to right, just like reading a text in Western writing. Musical rhythms are indicated by means of a sequence of symbols on the horizontal axis.

Music and its symbols

To understand the system of notation, it is necessary to identify the properties of the 'material' to be represented, i.e. the sounds and the symbols that are associated with them:

– First of all, the presence and absence of sound should be considered. Sound and its articulations are the material, although as a contrast between the presence of sound, pauses (or rests) are also important.
– Sounds are a result of a movement and have a start and a finish.

The notes and pauses are signs that represent the presence and absence of sound. The symbol itself indicates its relative duration with respect to the other sounds and silences.

Notes

Musical notes make up a code that indicates the duration of the sound they represent. The elements that are combined to obtain this are:

– Notehead: small black or white oval.
– Stem: Vertical segment with the notehead at one end and the flag at the other (where applicable).
– Flags: small curved lines located on the stem at the opposite end of the notehead.

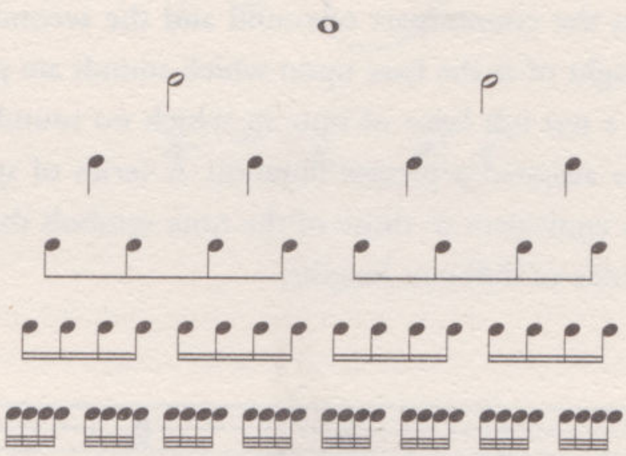
Relative duration of notes

Notes and pauses maintain a 'relative duration' independent of the speed at which the music is played. This speed at which they are played, and hence the 'real' duration of the notes in time is defined by a metronome mark, the rate of which is often given using a device of that name with an adjustable but constant speed.

As already noted, the relative duration of the notes is determined by the appearance of the head (black or white) and the presence or absence of a stem and flags.

Hence the notehead of minims and semibreves is white, whereas it is black for all the other notes. Additionally, with the exception of the semibreve, all have a stem. Quavers have one flag, semiquavers two, demisemiquavers three, and hemidemisemiquavers four. For each of the different notes, a relative duration is assigned with a number 2^n where n is between 0 and 6.

The sequence of notes from least to greatest duration is: semibreve, minim, crotchet, quaver, semiquaver, demisemiquaver and hemidemisemiquaver. The basic note is the semibreve, which is assigned the value 1. It is followed by the minim, which lasts for half the duration of a semibreve, meaning that the duration of the sound of a semibreve can be played using two minims. The duration of a minim contains two crotchets. In general, the duration of a note is half the previous note. The following diagram shows the relative duration of the notes, with the semibreve at the top of the pyramid and the hemidemisemiquavers at the base:



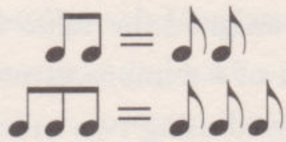
The following table gives the relative durations of each note:

N	2^n	Name	Note	Duration with respect to the semibreve
0	1	Semibreve	𝅝	1
1	2	Minim	𝅗𝅥	1/2
2	4	Crotchet	𝅘𝅥	1/4
3	8	Quaver	𝅘𝅥𝅮	1/8
4	16	Semiquaver	𝅘𝅥𝅯	1/16
5	32	Demisemiquaver	𝅘𝅥𝅰	1/32
6	64	Hemidemisemiquaver	𝅘𝅥𝅱	1/64

The number that characterises each note indicates the number of times it can be played in a sequence made up only of that note in order to replace a semibreve.

The ratio between the lengths of the notes is direct and transitive: if a minim is equivalent to two crotchets and a crotchet is equivalent to two semiquavers, a minim is equivalent to eight semiquavers.

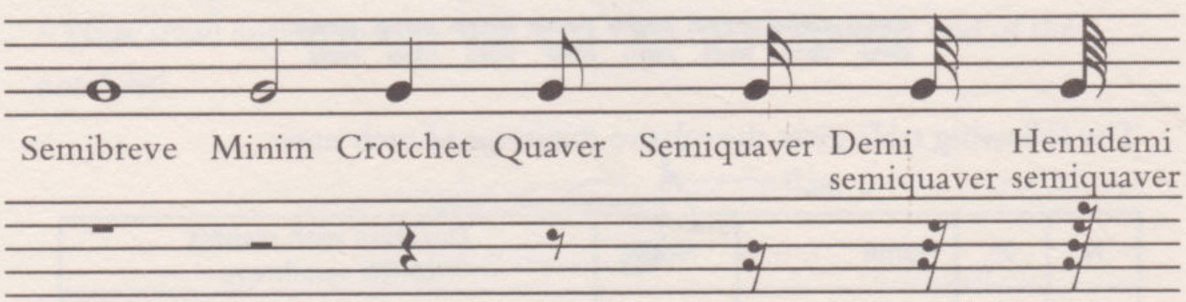
Quavers are often grouped by means of a beam that joins the flags in groups which generally correspond to the order established by a larger note, or is linked to the beat, as shown:



Rests

The rest, or pause, is the counterpart of sound and the second basic element of music. It can be thought of as the base upon which sounds are played, although in musical progression, a rest is a lapse of time in which no sound is played. Consequently, rest must be assigned a precise duration. A series of special symbols the duration of which is equivalent to those of the note symbols that are used to represent periods of silence of different lengths.

Notes



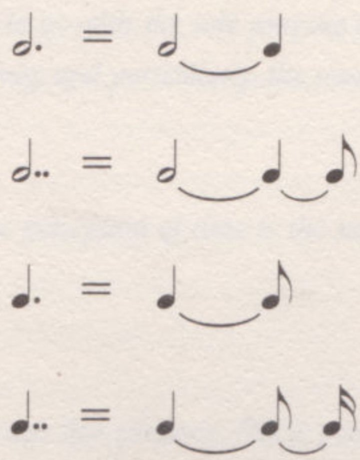
Rests

Dotted notes

There is often a need to increase the relative duration of a note (or rest). This is indicated by placing a small dot to the right of the notehead. Its presence indicates that the relative duration of the dotted note should be increased by 50%. Thus a dotted crotchet will have the same duration as a crotchet plus half of this note, or rather a quaver. The crotchet has a duration of 1/4 and the duration of

the quaver is half of this, or 1/8; hence the dotted crotchet will have a duration of 1/4 + 1/8 = 3/8, or three quavers.

There are also 'double-dotted' notes which indicate that the duration of the original note should be increased by 75%. In the case of a minim, the first dot increases its duration by a crotchet, and the second by a quaver. In the case of a crotchet, the dot extends the note by the equivalent of a quaver, and the double dot by the equivalent of a semiquaver:



Appendix II

A Second Look at the Role of Time in Music

Music is given to us with the sole purpose of establishing an order in things, including, and particularly, the coordination between man and time.
Igor Stravinsky

The perception of time is the source of all music and all rhythm.
Olivier Messiaen

We live in the moment. We are the present. All is present. We only know the past and the present. The present was, at best, imagined in the past. Perhaps in some way the present is conditioned by the past, despite never being anticipated.

We can step back to see a house in perspective or look at the path from afar. Time however, has us trapped. We cannot step back from time, we cannot stop it to allow us to think or rest. Yet in spite of this, humanity has been able to experience temporal processes. The method is simple: an event is planned in the future, we wait until the present reaches it and at that point it is recorded. A moment later, it reaches the past… and is recorded in memory.

The same thing happens with music: sound occurs in the present and is completed in memory. Music is impregnated in time, consciously or unconsciously.

Modality and tonality

There are at least two styles of music, or ways of understanding it – modal and tonal. They are distinguished from each other by the difference in how each develops its existence in time.

In the Western world, tonal music is the most common. The style was born during the Baroque period and was developed further between approximately 1600 and 1750. It is characterised by projecting itself forwards, towards the future. At each

point in the musical discourse of a tonal work, one clearly heard chord leads to the next chord. The harmonic tensions create the requirement to be resolved at a point of calm. The role played by the chord in this chaining of tensions and calm is referred to as the function of the chord.

From the Baroque period onwards, the tonal system underwent a permanent process of change through the periods of Western music which followed: Classical and Romantic. Even if tonality continues to dominate the majority of the music produced in the West, the highbrow work of the musical *avant garde* at the start of the 20th century began to move away from this in favour of a style of music without harmonic tensions, referred to as atonal or modal.

In the modal style, time can be interpreted in two different ways: on the one hand as eternity, the best known example of which is the Gregorian chant of medieval Europe, where there is no notion of past, present and future (i.e. time does not exist). The other concept of time is as the continuous present: the sound event occurs at each point without depending on conditions set by the previous one, nor does it need to condition the following point. Only the present matters. In addition to the highbrow music of the avant garde, a large part of Eastern music, certain South American folk music and bebop jazz conform to this second system.

The particular relationship between time and one style or another (tonal and modal) permits analogies with other forms of art: tonal music, could be associated with dance, and modal music on the other hand, with poetry.

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Index

Rhythm, Resonance and Harmony

The mathematics of music

A great mathematician once said that music was "the pleasure that the human mind experiences from counting without being aware of it." There are many fascinating connections between music and mathematics, from the relationship between harmony and numbers – an idea that has amazed mathematicians since the time of Pythagoras – to the ingenious techniques of repetition and translation used to stunning effect by legendary composers such as Bach and Mozart to compose their greatest masterpieces.